Chapter 299

Linear Regression and Correlation (Old Version)

Introduction

Linear Regression refers to a group of techniques for fitting and studying the straight-line relationship between two variables. Linear regression estimates the regression coefficients β_0 and β_1 in the equation

$$Y_j = \beta_0 + \beta_1 X_j + \varepsilon_j$$

where *X* is the independent variable, *Y* is the dependent variable, β_0 is the *Y intercept*, β_1 is the *slope*, and ε is the error.
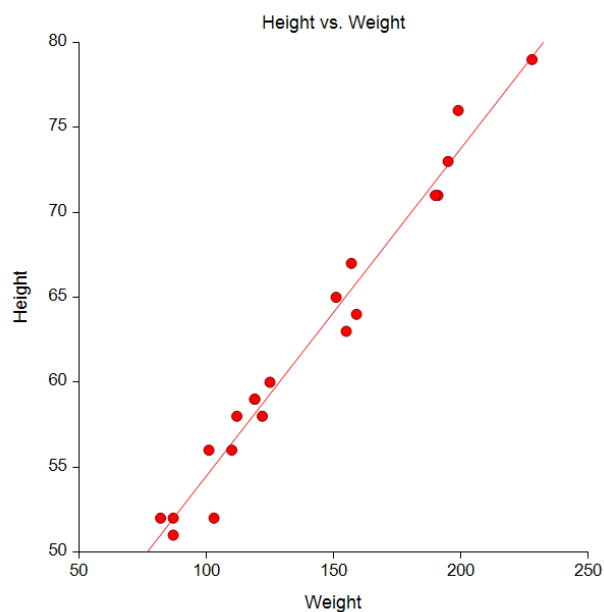
In order to calculate confidence intervals and hypothesis tests, it is assumed that the errors are independent and normally distributed with mean zero and variance σ^2.

Given a sample of *N* observations on *X* and *Y*, the method of least squares estimates β_0 and β_1 as well as various other quantities that describe the precision of the estimates and the goodness-of-fit of the straight line to the data. Since the estimated line will seldom fit the data exactly, a term for the discrepancy between the actual and fitted data values must be added. The equation then becomes

$$y_j = b_0 + b_1 x_j + e_j$$

$$= \hat{y}_j + e_j$$

where j is the observation (row) number, b_0 estimates β_0, b_1 estimates β_1, and e_j is the discrepancy between the actual data value y_j and the fitted value given by the regression equation, which is often referred to as $\hat{y}_j$. This discrepancy is usually referred to as the *residual*.

Note that the linear regression equation is a mathematical model describing the relationship between X and Y. In most cases, we do not believe that the model defines the exact relationship between the two variables. Rather, we use it as an approximation to the exact relationship. Part of the analysis will be to determine how close the approximation is.

Also note that the equation predicts Y from X. The value of Y depends on the value of X. The influence of all other variables on the value of Y is lumped into the residual.

Correlation

Once the intercept and slope have been estimated using least squares, various indices are studied to determine the reliability of these estimates. One of the most popular of these reliability indices is the *correlation coefficient*. The correlation coefficient, or simply the *correlation*, is an index that ranges from -1 to 1. When the value is near zero, there is no linear relationship. As the correlation gets closer to plus or minus one, the relationship is stronger. A value of one (or negative one) indicates a perfect linear relationship between two variables.

Actually, the strict interpretation of the correlation is different from that given in the last paragraph. The correlation is a parameter of the bivariate normal distribution. This distribution is used to describe the association between two variables. This association does not include a cause-and-effect statement. That is, the variables are not labeled as dependent and independent. One does not depend on the other. Rather, they are considered as two random variables that seem to vary together. The important point is that in linear regression, Y is assumed to be a random variable and X is assumed to be a fixed variable. In correlation analysis, both Y and X are assumed to be random variables.

Possible Uses of Linear Regression Analysis

Montgomery (1982) outlines the following four purposes for running a regression analysis.

Description

The analyst is seeking to find an equation that describes or summarizes the relationship between two variables. This purpose makes the fewest assumptions.

Coefficient Estimation

This is a popular reason for doing regression analysis. The analyst may have a theoretical relationship in mind, and the regression analysis will confirm this theory. Most likely, there is specific interest in the magnitudes and signs of the coefficients. Frequently, this purpose for regression overlaps with others.

Prediction

The prime concern here is to predict the response variable, such as sales, delivery time, efficiency, occupancy rate in a hospital, reaction yield in some chemical process, or strength of some metal. These predictions may be very crucial in planning, monitoring, or evaluating some process or system. There are many assumptions and qualifications that must be made in this case. For instance, you must not extrapolate beyond the range of the data. Also, interval estimates require that normality assumptions to hold.

Control

Regression models may be used for monitoring and controlling a system. For example, you might want to calibrate a measurement system or keep a response variable within certain guidelines. When a regression model is used for control purposes, the independent variable must be related to the dependent variable in a causal way. Furthermore, this functional relationship must continue over time. If it does not, continual modification of the model must occur.

Assumptions

The following assumptions must be considered when using linear regression analysis.

Linearity

Linear regression models the straight-line relationship between Y and X. Any curvilinear relationship is ignored. This assumption is most easily evaluated by using a scatter plot. This should be done early on in your analysis. Nonlinear patterns can also show up in residual plot. A lack of fit test is also provided.

Constant Variance

The variance of the residuals is assumed to be constant for all values of X. This assumption can be detected by plotting the residuals versus the independent variable. If these residual plots show a rectangular shape, we can assume constant variance. On the other hand, if a residual plot shows an increasing or decreasing wedge or bowtie shape, nonconstant variance (*heteroscedasticity*) exists and must be corrected.

The corrective action for nonconstant variance is to use weighted linear regression or to transform either Y or X in such a way that variance is more nearly constant. The most popular *variance stabilizing transformation* is the to take the logarithm of Y.

Special Causes

It is assumed that all special causes, outliers due to one-time situations, have been removed from the data. If not, they may cause nonconstant variance, nonnormality, or other problems with the regression model. The existence of outliers is detected by considering scatter plots of Y and X as well as the residuals versus X. Outliers show up as points that do not follow the general pattern.

Normality

When hypothesis tests and confidence limits are to be used, the residuals are assumed to follow the normal distribution.

Independence

The residuals are assumed to be uncorrelated with one another, which implies that the *Y's* are also uncorrelated. This assumption can be violated in two ways: model misspecification or time-sequenced data.

1. *Model misspecification.* If an important independent variable is omitted or if an incorrect functional form is used, the residuals may not be independent. The solution to this dilemma is to find the proper functional form or to include the proper independent variables and use multiple regression.

2. *Time-sequenced data.* Whenever regression analysis is performed on data taken over time, the residuals may be correlated. This correlation among residuals is called *serial correlation*. Positive serial correlation means that the residual in time period *j* tends to have the same sign as the residual in time period (*j* - *k*), where *k* is the lag in time periods. On the other hand, negative serial correlation means that the residual in time period *j* tends to have the opposite sign as the residual in time period (*j* - *k*).

The presence of serial correlation among the residuals has several negative impacts.

1. The regression coefficients remain unbiased, but they are no longer efficient, i.e., minimum variance estimates.

2. With positive serial correlation, the mean square error may be seriously underestimated. The impact of this is that the standard errors are underestimated, the *t*-tests are inflated (show significance when there is none), and the confidence intervals are shorter than they should be.

3. Any hypothesis tests or confidence limits that require the use of the *t* or *F* distribution are invalid.

You could try to identify these serial correlation patterns informally, with the residual plots versus time. A better analytical way would be to use the Durbin-Watson test to assess the amount of serial correlation.

Technical Details

Regression Analysis

This section presents the technical details of least squares regression analysis using a mixture of summation and matrix notation. Because this module also calculates weighted linear regression, the formulas will include the weights, w_j. When weights are not used, the w_j are set to one.

Define the following vectors and matrices.

$$Y = \begin{bmatrix} y_1 \\ \vdots \\ y_j \\ \vdots \\ y_N \end{bmatrix}, X = \begin{bmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_j \\ \vdots & \vdots \\ 1 & x_N \end{bmatrix}, e = \begin{bmatrix} e_1 \\ \vdots \\ e_j \\ \vdots \\ e_N \end{bmatrix}, 1 = \begin{bmatrix} 1 \\ \vdots \\ 1 \\ \vdots \\ 1 \end{bmatrix}, b = \begin{bmatrix} b_0 \\ b_1 \end{bmatrix}$$

$$W = \begin{bmatrix} w_1 & 0 & 0 & \cdots & 0 \\ 0 & \ddots & 0 & 0 & \vdots \\ 0 & 0 & w_j & 0 & 0 \\ \vdots & 0 & 0 & \ddots & 0 \\ 0 & \cdots & 0 & 0 & w_N \end{bmatrix}$$

Least Squares

Using this notation, the least squares estimates are found using the equation.

$$\boldsymbol{b} = (\boldsymbol{X'WX})^{-1}\boldsymbol{X'WY}$$

Note that when the weights are not used, this reduces to

$$\boldsymbol{b} = (\boldsymbol{X'X})^{-1}\boldsymbol{X'Y}$$

The predicted values of the dependent variable are given by

$$\widehat{\boldsymbol{Y}} = \boldsymbol{b'X}$$

The residuals are calculated using

$$\boldsymbol{e} = \boldsymbol{Y} - \widehat{\boldsymbol{Y}}$$

Estimated Variances

An estimate of the variance of the residuals is computed using

$$s^2 = \frac{\boldsymbol{e'We}}{N-2}$$

An estimate of the variance of the regression coefficients is calculated using

$$V\begin{pmatrix} b_0 \\ b_1 \end{pmatrix} = \begin{pmatrix} s_{b_0}^2 & s_{b_0 b_1} \\ s_{b_0 b_1} & s_{b_1}^2 \end{pmatrix}$$

$$= s^2 (\boldsymbol{X'WX})^{-1}$$

An estimate of the variance of the predicted mean of Y at a specific value of X, say X_0, is given by

$$s_{Y_m|X_0}^2 = s^2 (1, X_0)(\boldsymbol{X'WX})^{-1}\begin{pmatrix} 1 \\ X_0 \end{pmatrix}$$

An estimate of the variance of the predicted value of Y for an individual for a specific value of X, say X_0, is given by

$$s_{Y_I|X_0}^2 = s^2 + s_{Y_m|X_0}^2$$

Hypothesis Tests of the Intercept and Slope

Using these variance estimates and assuming the residuals are normally distributed, hypothesis tests may be constructed using the Student's t distribution with N - 2 degrees of freedom using

$$t_{b_0} = \frac{b_0 - B_0}{s_{b_0}}$$

and

$$t_{b_1} = \frac{b_1 - B_1}{s_{b_1}}$$

Usually, the hypothesized values of B_0 and B_1 are zero, but this does not have to be the case.

Confidence Intervals of the Intercept and Slope

A $100(1 - \alpha)\%$ confidence interval for the intercept, β_0, is given by

$$b_0 \pm t_{1-\alpha/2, N-2} s_{b_0}$$

A $100(1 - \alpha)\%$ confidence interval for the slope, β_1, is given by

$$b_1 \pm t_{1-\alpha/2, N-2} s_{b_1}$$

Confidence Interval of Y for Given X

A $100(1 - \alpha)\%$ confidence interval for the mean of Y at a specific value of X, say X_0, is given by

$$b_0 + b_1 X_0 \pm t_{1-\alpha/2, N-2} s_{Y_m|X_0}$$

Note that this confidence interval assumes that the sample size at X is N.

A $100(1 - \alpha)\%$ prediction interval for the value of Y for an individual at a specific value of X, say X_0, is given by

$$b_0 + b_1 X_0 \pm t_{1-\alpha/2, N-2} s_{Y_I|X_0}$$

Working-Hotelling Confidence Band for the Mean of Y

A $100(1 - \alpha)\%$ simultaneous confidence band for the mean of Y at all values of X is given by

$$b_0 + b_1 X \pm s_{Y_m|X} \sqrt{2 F_{1-\alpha, 2, N-2}}$$

This confidence band applies to all possible values of X. The confidence coefficient, $100(1 - \alpha)\%$, is the percent of a long series of samples for which this band covers the entire line for all values of X from negativity infinity to positive infinity.

Confidence Interval of X for Given Y

This type of analysis is called *inverse prediction* or *calibration*. A $100(1 - \alpha)\%$ confidence interval for the mean value of *X* for a given value of *Y* is calculated as follows. First, calculate *X* from *Y* using

$$\hat{X} = \frac{Y - b_0}{b_1}$$

Then, calculate the interval using

$$\frac{(\hat{X} - g\bar{X}) \pm A \sqrt{\frac{(1 - g)}{N} + \frac{(\hat{X} - \bar{X})^2}{\sum_{j=1}^{N} w_j (X_j - \bar{X})}}}{1 - g}$$

where

$$A = \frac{t_{1 - \alpha/2, N-2} s}{b_1}$$

$$g = \frac{A^2}{\sum_{j=1}^{N} w_j (X_j - \bar{X})}$$

A $100(1 - \alpha)\%$ confidence interval for an individual value of *X* for a given value of *Y* is

$$\frac{(\hat{X} - g\bar{X}) \pm A \sqrt{\frac{(N + 1)(1 - g)}{N} + \frac{(\hat{X} - \bar{X})^2}{\sum_{j=1}^{N} w_j (X_j - \bar{X})}}}{1 - g}$$

R-Squared (Percent of Variation Explained)

Several measures of the goodness-of-fit of the regression model to the data have been proposed, but by far the most popular is R^2. R^2 is the square of the correlation coefficient. It is the proportion of the variation in *Y* that is accounted by the variation in *X*. R^2 varies between zero (no linear relationship) and one (perfect linear relationship).

R^2, officially known as the coefficient of determination, is defined as the sum of squares due to the regression divided by the adjusted total sum of squares of *Y*. The formula for R^2 is

$$R^2 = 1 - \left(\frac{\boldsymbol{e'We}}{\boldsymbol{Y'WY} - \frac{(\boldsymbol{1'WY})^2}{\boldsymbol{1'W1}}} \right)$$

$$= \frac{SS_{Model}}{SS_{Total}}$$

R^2 is probably the most popular measure of how well a regression model fits the data. R^2 may be defined either as a ratio or a percentage. Since we use the ratio form, its values range from zero to one. A value of R^2 near zero indicates no linear relationship, while a value near one indicates a perfect linear fit. Although

popular, R^2 should not be used indiscriminately or interpreted without scatter plot support. Following are some qualifications on its interpretation:

1. *Additional independent variables*. It is possible to increase R^2 by adding more independent variables, but the additional independent variables may actually cause an increase in the mean square error, an unfavorable situation. This usually happens when the sample size is small.

2. *Range of the independent variable*. R^2 is influenced by the range of the independent variable. R^2 increases as the range of X increases and decreases as the range of the X decreases.

3. *Slope magnitudes*. R^2 does not measure the magnitude of the slopes.

4. *Linearity*. R^2 does not measure the appropriateness of a linear model. It measures the strength of the linear component of the model. Suppose the relationship between X and Y was a perfect circle. Although there is a perfect relationship between the variables, the R^2 value would be zero.

5. *Predictability*. A large R^2 does not necessarily mean high predictability, nor does a low R^2 necessarily mean poor predictability.

6. *No-intercept model*. The definition of R^2 assumes that there is an intercept in the regression model. When the intercept is left out of the model, the definition of R^2 changes dramatically. The fact that your R^2 value increases when you remove the intercept from the regression model does not reflect an increase in the goodness of fit. Rather, it reflects a change in the underlying definition of R^2.

7. *Sample size*. R^2 is highly sensitive to the number of observations. The smaller the sample size, the larger its value.

Rbar-Squared (Adjusted R-Squared)

R^2 varies directly with N, the sample size. In fact, when $N = 2$, $R^2 = 1$. Because R^2 is so closely tied to the sample size, an adjusted R^2 value, called $\bar{R}^2$, has been developed. $\bar{R}^2$ was developed to minimize the impact of sample size. The formula for $\bar{R}^2$ is

$$\bar{R}^2 = 1 - \left[\frac{\left(N - (p-1)\right)(1 - R^2)}{N - p} \right]$$

where p is 2 if the intercept is included in the model and 1 if not.

Probability Ellipse

When both variables are random variables and they follow the bivariate normal distribution, it is possible to construct a probability ellipse for them (see Jackson (1991) page 342). The equation of the $100(1 - \alpha)\%$ probability ellipse is given by those values of X and Y that are solutions of

$$T_{2,N-2,\alpha}^2 = \frac{s_{YY}s_{XX}}{s_{YY}s_{XX} - s_{XY}^2} \left[\frac{(X - \bar{X})^2}{s_{XX}} + \frac{(Y - \bar{Y})^2}{s_{YY}} - \frac{2s_{XY}(X - \bar{X})(Y - \bar{Y})}{s_{XX}s_{YY}} \right]$$

Orthogonal Regression Line

The least squares estimates discussed above minimize the sum of the squared distances between the *Y*'s and their predicted values. In some situations, both variables are random variables and it is arbitrary which is designated as the dependent variable and which is the independent variable. When the choice of which variable is the dependent variable is arbitrary, you may want to use the *orthogonal regression line* rather than the least squares regression line. The orthogonal regression line minimizes the sum of the squared perpendicular distances from each observation to the regression line. The orthogonal regression line is the first principal component when a principal components analysis is run on the two variables.

Jackson (1991) page 343 gives a formula for computing the orthogonal regression line without computing a principal components analysis. The slope is given by

$$b_{ortho,1} = \frac{s_{YY} - s_{XX} + \sqrt{s_{YY} - s_{XX} + 4s_{XY}^2}}{2s_{XY}}$$

where

$$s_{XY} = \frac{\sum_{j=1}^{N} w_j(X_j - \bar{X})(Y_j - \bar{Y})}{N - 1}$$

The estimate of the intercept is then computed using

$$b_{ortho,y} = \bar{Y} - b_{ortho,1}\bar{X}$$

Although Jackson gives formulas for a confidence interval on the slope and intercept, we do not provide them in **NCSS** because their properties are not well understood and require certain bivariate normal assumptions. Instead, **NCSS** provides bootstrap confidence intervals for the slope and intercept.

The Correlation Coefficient

The correlation coefficient can be interpreted in several ways. Here are some of the interpretations.

1. If both *Y* and *X* are standardized by subtracting their means and dividing by their standard deviations, the correlation is the slope of the regression of the standardized *Y* on the standardized *X*.

2. The correlation is the standardized covariance between *Y* and *X*.

3. The correlation is the geometric average of the slopes of the regressions of *Y* on *X* and of *X* on *Y*.

4. The correlation is the square root of *R*-squared, using the sign from the slope of the regression of *Y* on *X*.

The corresponding formulas for the calculation of the correlation coefficient are

$$r = \frac{\sum_{j=1}^{N} w_j (X_j - \bar{X})(Y_j - \bar{Y})}{\sqrt{\left[\sum_{j=1}^{N} w_j (X_j - \bar{X})^2\right]\left[\sum_{j=1}^{N} w_j (Y_j - \bar{Y})^2\right]}}$$

$$= \frac{s_{XY}}{\sqrt{s_{XX} s_{YY}}}$$

$$= \pm\sqrt{b_{YX} b_{XY}}$$

$$= \text{sign}(b_{YX})\sqrt{R^2}$$

where s_{XY} is the covariance between X and Y, b_{XY} is the slope from the regression of X on Y, and b_{YX} is the slope from the regression of Y on X. s_{XY} is calculated using the formula

$$s_{XY} = \frac{\sum_{j=1}^{N} w_j (X_j - \bar{X})(Y_j - \bar{Y})}{N - 1}$$

The *population correlation coefficient*, ρ, is defined for two random variables, U and W, as follows

$$\rho = \frac{\sigma_{UW}}{\sqrt{\sigma_U^2 \sigma_W^2}}$$

$$= \frac{E[(U - \mu_U)(W - \mu_W)]}{\sqrt{Var(U)Var(W)}}$$

Note that this definition does not refer to one variable as dependent and the other as independent. Rather, it simply refers to two random variables.

Facts about the Correlation Coefficient

The correlation coefficient has the following characteristics.

1. The range of r is between -1 and 1, inclusive.

2. If r = 1, the observations fall on a straight line with positive slope.

3. If r = -1, the observations fall on a straight line with negative slope.

4. If r = 0, there is no linear relationship between the two variables.

5. r is a measure of the linear (straight-line) association between two variables.

6. The value of r is unchanged if either X or Y is multiplied by a constant or if a constant is added.

7. The physical meaning of r is mathematically abstract and may not be very helpful. However, we provide it for completeness. The correlation is the cosine of the angle formed by the intersection of two vectors in N-dimensional space. The components of the first vector are the values of X while the components of the second vector are the corresponding values of Y. These components are arranged so that the first dimension corresponds to the first observation, the second dimension corresponds to the second observation, and so on.

Hypothesis Tests for the Correlation

You may be interested in testing hypotheses about the population correlation coefficient, such as $\rho = \rho_0$. When $\rho_0 = 0$, the test is identical to the t-test used to test the hypothesis that the slope is zero. The test statistic is calculated using

$$t_{N-2} = \frac{r}{\sqrt{\dfrac{1 - r^2}{N - 2}}}$$

However, when $\rho_0 \neq 0$, the test is different from the corresponding test that the slope is a specified, nonzero, value.

NCSS provides two methods for testing whether the correlation is equal to a specified, nonzero, value.

Method 1. This method uses the distribution of the correlation coefficient. Under the null hypothesis that $\rho = \rho_0$ and using the distribution of the sample correlation coefficient, the likelihood of obtaining the sample correlation coefficient, r, can be computed. This likelihood is the statistical significance of the test. This method requires the assumption that the two variables follow the bivariate normal distribution.

Method 2. This method uses the fact that Fisher's z transformation, given by

$$F(r) = \frac{1}{2} \ln\left(\frac{1 + r}{1 - r}\right)$$

is closely approximated by a normal distribution with mean

$$\frac{1}{2} \ln\left(\frac{1 + \rho}{1 - \rho}\right)$$

and variance

$$\frac{1}{N - 3}$$

To test the hypothesis that $\rho = \rho_0$, you calculate z using

$$z = \frac{F(r) - F(\rho_0)}{\sqrt{\dfrac{1}{N - 3}}}$$

$$= \frac{\ln\left(\dfrac{1 + r}{1 - r}\right) - \ln\left(\dfrac{1 - \rho_0}{1 - \rho_0}\right)}{2\sqrt{\dfrac{1}{N - 3}}}$$

and use the fact that z is approximately distributed as the standard normal distribution with mean equal to zero and variance equal to one. This method requires two assumptions. First, that the two variables follow the bivariate normal distribution. Second, that the distribution of z is approximated by the standard normal distribution.

This method has become popular because it uses the commonly available normal distribution rather than the obscure correlation distribution. However, because it makes an additional assumption, it is not as accurate as is method 1. In fact, we have included in for completeness, but recommend the use of Method 1.

Confidence Intervals for the Correlation

A $100(1 - \alpha)\%$ confidence interval for ρ may be constructed using either of the two hypothesis methods described above. The confidence interval is calculated by finding, either directly using Method 2 or by a search using Method 1, all those values of ρ_0 for which the hypothesis test is not rejected. This set of values becomes the confidence interval.

Be careful not to make the common mistake in assuming that this confidence interval is related to a transformation of the confidence interval on the slope β_1. The two confidence intervals are not simple transformations of each other.

Spearman Rank Correlation Coefficient

The *Spearman rank correlation coefficient* is a popular nonparametric analog of the usual correlation coefficient. This statistic is calculated by replacing the data values with their ranks and calculating the correlation coefficient of the ranks. Tied values are replaced with the average rank of the ties. This coefficient is really a measure of association rather than correlation, since the ranks are unchanged by a monotonic transformation of the original data.

When *N* is greater than 10, the distribution of the Spearman rank correlation coefficient can be approximated by the distribution of the regular correlation coefficient.

Note that when weights are specified, the calculation of the Spearman rank correlation coefficient uses the weights.

Smoothing with Loess

The *loess* (locally weighted regression scatter plot smoothing) method is used to obtain a smooth curve representing the relationship between *X* and *Y.* Unlike linear regression, loess does not have a simple mathematical model. Rather, it is an algorithm that, given a value of *X*, computes an appropriate value of *Y*. The algorithm was designed so that the loess curve travels through the middle of the data, summarizing the relationship between *X* and *Y*.

The loess algorithm works as follows.

1. Select a value for *X*. Call it *X*0.

2. Select a neighborhood of points close to *X*0.

3. Fit a weighted regression of *Y* on *X* using only the points in this neighborhood. In the regression, the weights are inversely proportional to the distance between *X* and *X*0.

4. To make the procedure robust to outliers, a robust regression may be substituted for the weighted regression in step 3. This robust procedure modifies the weights so that observations with large residuals receive smaller weights.

5. Use the regression coefficients from the weighted regression in step 3 to obtain a predicted value for *Y* at *X*0.

6. Repeat steps 1 - 5 for a set of *X*'s between the minimum and maximum of *X*.

Mathematical Details of Loess

This section presents the mathematical details of the loess method of scatter plot smoothing. Note that implicit in the discussion below is the assumption that *Y* is the dependent variable and *X* is the independent variable.

Loess gives the value of *Y* for a given value of *X*, say *X*0. For each observation, define the distance between *X* and *X*0 as

$$d_j = |X_j - X0|$$

Let *q* be the number of observations in the neighborhood of *X*0. Define *q* as [*fN*] where *f* is the user-supplied fraction of the sample. Here, [*Z*] is the largest integer in *Z*. Often *f* = 0.40 is a good choice. The neighborhood is defined as the observations with the *q* smallest values of d_j. Define d_q as the largest distance in the neighborhood of observations close to *X*0.

The tricube weight function is defined as

$$T(u) = \begin{cases} (1 - |u|^3)^3 & |u| < 1 \\ 0 & |u| \geq 1 \end{cases}$$

The weight for each observation is defined as

$$w_j = T\left(\frac{|X_j - X0|}{d_q}\right)$$

The weighted regression for *X*0 is defined by the value of *b*0, *b*1, and *b*2 that minimize the sum of squares

$$\sum_{j=1}^{N} T\left(\frac{X_j - X0}{d_q}\right)\left(Y_j - b0 - b1(X_j) - b2(X_j)^2\right)^2$$

Note the if *b*2 is zero, a linear regression is fit. Otherwise, a quadratic regression is fit. The choice of linear or quadratic is an option in the procedure. The linear option is quicker, while the quadratic option fits peaks and valleys better. In most cases, there is little difference except at the extremes in the *X* space.

Once *b*0, *b*1, and *b*2 have be estimated using weighted least squares, the loess value is computed using

$$\hat{Y}_{loess}(X0) = b0 - b1(X0) - b2(X0)^2$$

Note that a separate weighted regression must be run for each value of *X*0.

Robust Loess

Outliers often have a large impact on least squares impact. A robust weighted regression procedure may be used to lessen the influence of outliers on the loess curve. This is done as follows.

The q loess residuals are computed using the loess regression coefficients using the formula

$$r_j = Y_j - \hat{Y}_{loess}(X_j)$$

New weights are defined as

$$w_j = w_{last,j} B\left(\frac{|r_j|}{6M}\right)$$

where $w_{last,j}$ is the previous weight for this observation, M is the median of the q absolute values of the residuals, and $B(u)$ is the bisquare weight function defined as

$$B(u) = \begin{cases} (1 - u^2)^2 & |u| < 1 \\ 0 & |u| \geq 1 \end{cases}$$

This robust procedure may be iterated up to five items, but we have seen little difference in the appearance of the loess curve after two iterations.

Note that it is not always necessary to create the robust weights. If you are not going to remove the outliers from your final results, you probably should not remove them from the loess curve by setting the number of robust iterations to zero.

Testing Assumptions Using Residual Diagnostics

Evaluating the amount of departure in your data from each linear regression assumption is necessary to see if any remedial action is necessary before the fitted results can be used. First, the types of plots and statistical analyses the are used to evaluate each assumption will be given. Second, each of the diagnostic values will be defined.

Notation – Use of (j) and p

Several of these residual diagnostic statistics are based on the concept of studying what happens to various aspects of the regression analysis when each row is removed from the analysis. In what follows, we use the notation (*j*) to mean that observation *j* has been omitted from the analysis. Thus, *b(j)* means the value of *b* calculated without using observation *j*.

Some of the formulas depend on whether the intercept is fitted or not. We use *p* to indicate the number of regression parameters. When the intercept is fit, *p* will be two. Otherwise, *p* will be one.

1 – No Outliers

Outliers are observations that are poorly fit by the regression model. If outliers are influential, they will cause serious distortions in the regression calculations. Once an observation has been determined to be an outlier, it must be checked to see if it resulted from a mistake. If so, it must be corrected or omitted. However, if no mistake can be found, the outlier should not be discarded just because it is an outlier. Many scientific discoveries have been made because outliers, data points that were different from the norm, were studied more closely. Besides being caused by simple data-entry mistakes, outliers often suggest the presence of an important independent variable that has been ignored.

Outliers are easy to spot on bar charts or box plots of the residuals and RStudent. RStudent is the preferred statistic for finding outliers because each observation is omitted from the calculation making it less likely that the outlier can mask its presence. Scatter plots of the residuals and RStudent against the *X* variable are also helpful because they may show other problems as well.

2 – Linear Regression Function - No Curvature

The relationship between *Y* and *X* is assumed to be linear (straight-line). No mechanism for curvature is included in the model. Although a scatter plot of *Y* versus *X* can show curvature in the relationship, the best diagnostic tool is the scatter plot of the residual versus *X*. If curvature is detected, the model must be modified to account for the curvature. This may mean adding quadratic terms, taking logarithms of *Y* or *X,* or some other appropriate transformation.

Loess Curve

A loess curve should be plotted between *X* and *Y* to see if any curvature is present.

Lack of Fit Test

When the data include repeat observations at one or more *X* values (*replicates*), the adequacy of the linear model can be evaluated numerically by performing a *lack of fit* test. This test procedure detects nonlinearities.

The lack of fit test is constructed as follows. First, the sum of squares for error is partitioned into two quantities: *lack of fit* and *pure error*. The pure error sum of squares is found by considering only those observations that are replicates. The *X* values are treated as the levels of the factor in a one-way analysis of variance. The sum of squares error from this analysis measures the underlying variation in *Y* that occurs when the value of *X* is held constant. Thus, it is called *pure error*. When the pure error sum of squares is subtracted from the error sum of squares of the linear regression, the result is measure of the amount of nonlinearity in the data. An *F*-ratio can be constructed from these two values that will test the statistical significance of the lack of fit. The *F*-ratio is constructed using the following equation.

$$F_{DF1,DF2} = \frac{\dfrac{SS_{Lack\ of\ fit}}{DF1}}{\dfrac{SS_{Pure\ Error}}{DF2}}$$

where *DF*2 is the degrees of freedom for the error term in the one-way analysis of variance and *DF*1 is *N* - *DF*2 - 2.

3 – Constant Variance

The errors are assumed to have constant variance across all values of *X*. If there are a lot of data (N > 100), nonconstant variance can be detected on a scatter plot of the residuals versus *X*. However, the most direct diagnostic tool to evaluate this assumption is a scatter plot of the absolute values of the residuals versus *X*. Often, the assumption is violated because the variance increases with *X*. This will show up as a 'megaphone' pattern to this plot.

When nonconstant variance is detected, a variance-stabilizing transformation such as the square-root or logarithm may be used. However, the best solution is probably to use weighted regression, with weights inversely proportional to the magnitude of the residuals.

Modified Levene Test

The *modified Levene test* can be used to evaluate the validity of the assumption of constant variance. It has been shown to be reliable even when the residuals do not follow a normal distribution.

The test is constructed by grouping the residuals according to the values of *X*. The number of groups is arbitrary, but usually, two groups are used. In this case, the absolute residuals of observations with low values of *X* are compared against those with high values of *X*. If the variability is constant, the variability in these two groups of residuals should be equal. The test is computed using the formula

$$L = \frac{\bar{d}_1 - \bar{d}_2}{s_L \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

where

$$s_L = \sqrt{\frac{\sum_{j=1}^{n_1}(d_{j1} - \bar{d}_1) + \sum_{j=1}^{n_2}(d_{j2} - \bar{d}_2)}{n_1 + n_2 - 2}}$$

$$d_{j1} = |e_{j1} - \tilde{e}_1|$$

$$d_{j2} = |e_{j2} - \tilde{e}_2|$$

and $\tilde{e}_1$ is the median of the group of residuals for low values of *X* and $\tilde{e}_2$ is the median of the group of residuals for high values of *X*. The test statistic *L* is approximately distributed as a *t* statistic with *N* - 2 degrees of freedom.

4 – Independent Errors

The *Y*'s, and thus the errors, are assumed to be independent. This assumption is usually ignored unless there is a reason to think that it has been violated, such as when the observations were taken across time. An easy way to evaluate this assumption is a scatter plot of the residuals versus their sequence number (assuming that the data are arranged in time sequence order). This plot should show a relative random pattern.

The Durbin-Watson statistic is used as a formal test for the presence of first-order serial correlation. A more comprehensive method of evaluation is to look at the autocorrelations of the residuals at various lags. Large autocorrelations are found by testing each using Fisher's *z* transformation. Although Fisher's *z* transformation is only approximate in the case of autocorrelations, it does provide a reasonable measuring stick with which to judge the size of the autocorrelations.

If independence is violated, confidence intervals and hypothesis tests are erroneous. Some remedial method that accounts for the lack of independence must be adopted, such as using first differences or the Cochrane-Orcutt procedure.

Durbin-Watson Test

The Durbin-Watson test is often used to test for positive or negative, first-order, serial correlation. It is calculated as follows

$$DW = \frac{\sum_{j=2}^{N}(e_j - e_{j-1})^2}{\sum_{j=1}^{N} e_j^2}$$

The distribution of this test is difficult because it involves the X values. Originally, Durbin-Watson (1950, 1951) gave a pair of bounds to be used. However, there is a large range of 'inclusion' found when using these bounds. Instead of using these bounds, we calculate the exact probability using the beta distribution approximation suggested by Durbin-Watson (1951). This approximation has been shown to be accurate to three decimal places in most cases which is all that are needed for practical work.

5 – Normality of Residuals

The residuals are assumed to follow the normal probability distribution with zero mean and constant variance. This can be evaluated using a normal probability plot of the residuals. Also, normality tests are used to evaluate this assumption. The most popular of the five normality tests provided is the Shapiro-Wilk test.

Unfortunately, a breakdown in any of the other assumptions results in a departure from this assumption as well. Hence, you should investigate the other assumptions first, leaving this assumption until last.

Influential Observations

Part of the evaluation of the assumptions includes an analysis to determine if any of the observations have an extra-large influence on the estimated regression coefficients, on the fit of the model, or on the value of Cook's distance. By looking at how much removing an observation changes the results, an observation's influence can be determined.

Five statistics are used to investigate influence. These are Hat diagonal, DFFITS, DFBETAS, Cook's D, and COVARATIO.

Definitions Used in Residual Diagnostics

Residual

The residual is the difference between the actual *Y* value and the *Y* value predicted by the estimated regression model. It is also called the *error*, the *deviate*, or the *discrepancy*.

$$e_j = y_j - \hat{y}_j$$

Although the true errors, ε_j, are assumed to be independent, the computed residuals, e_j, are not. Although the lack of independence among the residuals is a concern in developing theoretical tests, it is not a concern on the plots and graphs.

The variance of the ε_j is σ^2. However, the variance of the e_j is not σ^2. In vector notation, the covariance matrix of $\boldsymbol{e}$ is given by

$$V(\boldsymbol{e}) = \sigma^2 \left(\boldsymbol{I} - \boldsymbol{W}^{\frac{1}{2}} \boldsymbol{X} (\boldsymbol{X'WX})^{-1} \boldsymbol{X'W}^{\frac{1}{2}} \right)$$

$$= \sigma^2 (\boldsymbol{I} - \boldsymbol{H})$$

The matrix **H** is called the *hat matrix* since it puts the 'hat' on *y* as is shown in the unweighted case.

$$\hat{Y} = \boldsymbol{Xb}$$

$$= \boldsymbol{X}(\boldsymbol{X'X})^{-1} \boldsymbol{X'Y}$$

$$= \boldsymbol{HY}$$

Hence, the variance of e_j is given by

$$V(e_j) = \sigma^2 (1 - h_{jj})$$

where h_{jj} is the jth diagonal element of **H**. This variance is estimated using

$$\hat{V}(e_j) = s^2 (1 - h_{jj})$$

Hat Diagonal

The hat diagonal, h_{jj}, is the jth diagonal element of the hat matrix, **H**, where

$$H = W^{\frac{1}{2}}X(X'WX)^{-1}X'W^{\frac{1}{2}}$$

H captures an observation's remoteness in the *X*-space. Some authors refer to the hat diagonal as a measure of *leverage* in the *X*-space. As a rule of thumb, hat diagonals greater than 4/*N* are considered influential and are called high-leverage observations.

Note that a high-leverage observation is not a bad observation. Rather, high-leverage observations exert extra influence on the final results, so care should be taken to ensure that they are correct. You should not delete an observation just because it has a high-influence. However, when you interpret the regression equation, you should bear in mind that the results may be due to a few, high-leverage observations.

Standardized Residual

As shown above, the variance of the observed residuals is not constant. This makes comparisons among the residuals difficult. One solution is to standardize the residuals by dividing by their standard deviations. This will give a set of residuals with constant variance.

The formula for this residual is

$$r_j = \frac{e_j}{s\sqrt{1 - h_{jj}}}$$

s(j) or MSEi

This is the value of the mean squared error calculated without observation *j*. The formula for *s*(*j*) is given by

$$s(j)^2 = \frac{1}{N - p - 1} \sum_{i=1, i \neq j}^{N} w_i(y_i - x_i b(j))$$

$$= \frac{(N - p)s^2 - \dfrac{w_j e_j^2}{1 - h_{jj}}}{N - p - 1}$$

RStudent

Rstudent is similar to the studentized residual. The difference is the *s*(*j*) is used rather than *s* in the denominator. The quantity *s*(*j*) is calculated using the same formula as *s*, except that observation *j* is omitted. The hope is that be excluding this observation, a better estimate of σ² will be obtained. Some statisticians refer to these as the *studentized deleted residuals*.

$$t_j = \frac{e_j}{s(j)\sqrt{1 - h_{jj}}}$$

If the regression assumptions of normality are valid, a single value of the RStudent has a *t* distribution with *N* - 2 degrees of freedom. It is reasonable to consider |RStudent| > 2 as outliers.

DFFITS

DFFITS is the standardized difference between the predicted value with and without that observation. The formula for *DFFITS* is

$$DFFITS_j = \frac{\hat{y}_j - \hat{y}_j(j)}{s(j)\sqrt{h_{jj}}}$$

$$= t_j \sqrt{\frac{h_{jj}}{1 - h_{jj}}}$$

The values of $\hat{y}_j(j)$ and $s^2(j)$ are found by removing observation j before the doing the calculations. It represents the number of estimated standard errors that the fitted value changes if the jth observation is omitted from the data set. If $|DFFITS| > 1$, the observation should be considered to be influential with regards to prediction.

Cook's D

The DFFITS statistic attempts to measure the influence of a single observation on its fitted value. Cook's distance (Cook's *D*) attempts to measure the influence each observation on all *N* fitted values. The formula for Cook's *D* is

$$D_j = \frac{\sum_{i=1}^{N} w_j[\hat{y}_j - \hat{y}_j(i)]^2}{ps^2}$$

The $\hat{y}_j(i)$ are found by removing observation i before the calculations. Rather than go to all the time of recalculating the regression coefficients *N* times, we use the following approximation

$$D_j = \frac{w_j e_j^2 h_{jj}}{ps^2(1 - h_{jj})^2}$$

This approximation is exact when no weight variable is used.

A Cook's *D* value greater than one indicates an observation that has large influence. Some statisticians have suggested that a better cutoff value is 4 / (*N* - 2).

CovRatio

This diagnostic flags observations that have a major impact on the generalized variance of the regression coefficients. A value exceeding 1.0 implies that the ith observation provides an improvement, i.e., a reduction in the generalized variance of the coefficients. A value of CovRatio less than 1.0 flags an observation that increases the estimated generalized variance. This is not a favorable condition.

The general formula for the CovRatio is

$$CovRatio_j = \frac{\det\left[s(j)^2(X(j)'WX(j))^{-1}\right]}{\det[s^2(X'WX)^{-1}]}$$

$$= \frac{1}{1 - h_{jj}}\left[\frac{s(j)^2}{s^2}\right]^p$$

where p = 2 if the intercept is fit or 1 if not.

Belsley, Kuh, and Welsch (1980) give the following guidelines for the CovRatio:

- If CovRatio > 1 + 3p / N then omitting this observation significantly damages the precision of at least some of the regression estimates.

- If CovRatio < 1 - 3p / N then omitting this observation significantly improves the precision of at least some of the regression estimates.

DFBETAS

The *DFBETAS* criterion measures the standardized change in a regression coefficient when an observation is omitted. The formula for this criterion is

$$DFBETAS_{kj} = \frac{b_k - b_k(j)}{s(j)\sqrt{c_{kk}}}$$

where c_{kk} is a diagonal element of the inverse matrix $(X'WX)^{-1}$.

Belsley, Kuh, and Welsch (1980) recommend using a cutoff of $2/\sqrt{N}$ when N is greater than 100. When N is less than 100, others have suggested using a cutoff of 1.0 or 2.0 for the absolute value of DFBETAS.

Press Value

PRESS is an acronym for prediction sum of squares. It was developed for use in variable selection to validate a regression model. To calculate *PRESS*, each observation is individually omitted. The remaining N - 1 observations are used to calculate a regression and estimate the value of the omitted observation. This is done N times, once for each observation. The difference between the actual Y value and the predicted Y with the observation deleted is called the prediction error or *PRESS* residual. The sum of the squared prediction errors is the *PRESS* value. The smaller *PRESS* is, the better the predictability of the model.

The formula for PRESS is

$$PRESS = \sum_{j=1}^{N} w_j[y_j - \hat{y}_j(j)]^2$$

Press R-Squared

The PRESS value above can be used to compute an R^2-like statistic, called *R2Predict*, which reflects the prediction ability of the model. This is a good way to validate the prediction of a regression model without selecting another sample or splitting your data. It is very possible to have a high R^2 and a very low *R2Predict*. When this occurs, it implies that the fitted model is data dependent. This *R2Predict* ranges from below zero to above one. When outside the range of zero to one, it is truncated to stay within this range.

$$R^2_{Predict} = 1 - \frac{PRESS}{SS_{Total}}$$

Sum |Press residuals|

This is the sum of the absolute value of the *PRESS* residuals or prediction errors. If a large value for the *PRESS* is due to one or a few large *PRESS* residuals, this statistic may be a more accurate way to evaluate predictability. This quantity is computed as

$$\sum |PRESS| = \sum_{j=1}^{N} w_j |y_j - \hat{y}_j(j)|$$

Bootstrapping

Bootstrapping was developed to provide standard errors and confidence intervals for regression coefficients and predicted values in situations in which the standard assumptions are not valid. In these nonstandard situations, bootstrapping is a viable alternative to the corrective action suggested earlier. The method is simple in concept, but it requires extensive computation time.

The bootstrap is simple to describe. You assume that your sample is actually the population and you draw *B* samples (*B* is over 1000) of size *N* from your original sample with replacement. With replacement means that each observation may be selected more than once. For each bootstrap sample, the regression results are computed and stored.

Suppose that you want the standard error and a confidence interval of the slope. The bootstrap sampling process has provided *B* estimates of the slope. The standard deviation of these *B* estimates of the slope is the bootstrap estimate of the standard error of the slope. The bootstrap confidence interval is found by arranging the *B* values in sorted order and selecting the appropriate percentiles from the list. For example, a 90% bootstrap confidence interval for the slope is given by fifth and ninety-fifth percentiles of the bootstrap slope values. The bootstrap method can be applied to many of the statistics that are computed in regression analysis.

The main assumption made when using the bootstrap method is that your sample approximates the population fairly well. Because of this assumption, bootstrapping does not work well for small samples in which there is little likelihood that the sample is representative of the population. Bootstrapping should only be used in medium to large samples.

When applied to linear regression, there are two types of bootstrapping that can be used. See Neter, Kutner, Nachtsheim, Wasserman (1996) page 430.

Modified Residuals

Davison and Hinkley (1999) page 279 recommend the use of a special rescaling of the residuals when bootstrapping to keep results unbiased. These modified residuals are calculated using

$$e_j^* = \frac{e_j}{\sqrt{\dfrac{1 - h_{jj}}{w_j}}} - \bar{e}^*$$

where

$$\bar{e}^* = \frac{\sum_{j=1}^{N} w_j \, e_j^*}{\sum_{j=1}^{N} w_j}$$

Bootstrap the Observations

The bootstrap samples are selected from the original sample of X and Y pairs. This method is appropriate for data in which both X and Y have been selected at random. That is, the X values were not predetermined, but came in as measurements just as the Y values.

An example of this situation would be if a population of individuals is sampled and both Y and X are measured on those individuals only after the sample is selected. That is, the value of X was not used in the selection of the sample.

Bootstrap the Residuals

The bootstrap samples are constructed using the modified residuals. In each bootstrap sample, the randomly sampled modified residuals are added to the original fitted values forming new values of Y. This method forces the original structure of the X values to be retained in every bootstrap sample.

This method is appropriate for data obtained from a designed experiment in which the values of X are preset by the experimental design.

Because the residuals are sampled and added back at random, the method must assume that the variance of the residuals is constant. **If the sizes of the residuals are proportional to X, this method should not be used.**

Bootstrap Prediction Intervals

Bootstrap confidence intervals for the mean of Y given X are generated from the bootstrap sample in the usual way. To calculate prediction intervals for the predicted value (not the mean) of Y given X requires a modification to the predicted value of Y to account for the variation of Y about its mean. This modification of the predicted Y values in the bootstrap sample, suggested by Davison and Hinkley, is as follows.

$$\hat{y}_+ = \hat{y} - x(b_1^* - b_1) + e_+^*$$

where e_+^* is a randomly selected modified residual. By adding the randomly sampled residual we have added an appropriate amount of variation to represent the variance of individual Y's about their mean value.

Randomization Test

Because of the strict assumptions that must be made when using this procedure to test hypotheses about the slope, **NCSS** also includes a randomization test as outlined by Edgington (1987). Randomization tests are becoming more and more popular as the speed of computers allows them to be computed in seconds rather than hours.

A randomization test is conducted by enumerating all possible permutations of the dependent variable while leaving the independent variable in the original order. The slope is calculated for each permutation and the number of permutations that result in a slope with a magnitude greater than or equal to the actual slope is counted. Dividing this count by the number of permutations tried gives the significance level of the test.

For even moderate sample sizes, the total number of permutations is in the trillions, so a Monte Carlo approach is used in which the permutations are found by random selection rather than complete enumeration. Edgington suggests that at least 1,000 permutations be selected. We suggest that this be increased to 10,000.

Data Structure

The data are entered as two variables. If weights or frequencies are available, they are entered separately in other variables. An example of data appropriate for this procedure is shown below. These data are the heights and weights of twenty individuals. The data are contained in the LINREG1 database. We suggest that you open this database now so that you can follow along with the examples.

LinReg1 Dataset (Subset)

Height	Weight
64	159
63	155
67	157
60	125
52	103
58	122
56	101
52	82
79	228
76	199
73	195

Missing Values

Rows with missing values in the variables being analyzed are ignored. If data are present on a row for all but the dependent variable, a predicted value and confidence limits are generated for that row.

Example 1 – Running a Linear Regression Analysis

This section presents an example of how to run a linear regression analysis of the data in the LinReg1 dataset. In this example, we will run a regression of *Height* on *Weight*. Predicted values of Height are wanted at Weight values equal to 90, 100, 150, 200, and 250.

This regression program outputs over thirty different reports and plots, many of which contain duplicate information. For the purposes of annotating the output, we will output all the reports. (Normally, you would only select a few of these reports.)

Setup

To run this example, complete the following steps:

1   Open the LinReg1 example dataset

- From the File menu of the NCSS Data window, select **Open Example Data**.
- Select **LinReg1** and click **OK**.

2   Specify the Linear Regression and Correlation procedure options

- Find and open the **Linear Regression and Correlation** procedure using the menus or the Procedure Navigator.
- The settings for this example are listed below and are stored in the **Example 1** settings file. To load these settings to the procedure window, click **Open Example Settings File** in the Help Center or File menu.

Variables Tab

Y: Dependent Variable(s) ....**Height**
X: Independent Variable ....**Weight**
Calculate Bootstrap C.I.'s ....**Checked**
Run Randomization Tests ....**Checked**

Resampling Tab

Bootstrap Samples (N) **1000**
Random Seed...**3118927** (for reproducibility)

Reports Tab

Predict Y at these X Values **90 100 150 200 250**
Select a Group of Reports and Plots ....**Display ALL reports & plots** (As mentioned above, normally you would only view a few of these reports, but all are selected here for the purpose of documenting each report.)

---

3   Run the procedure

- Click the **Run** button to perform the calculations and generate the output.

Linear Regression Plot Section

Linear Regression Plot Section
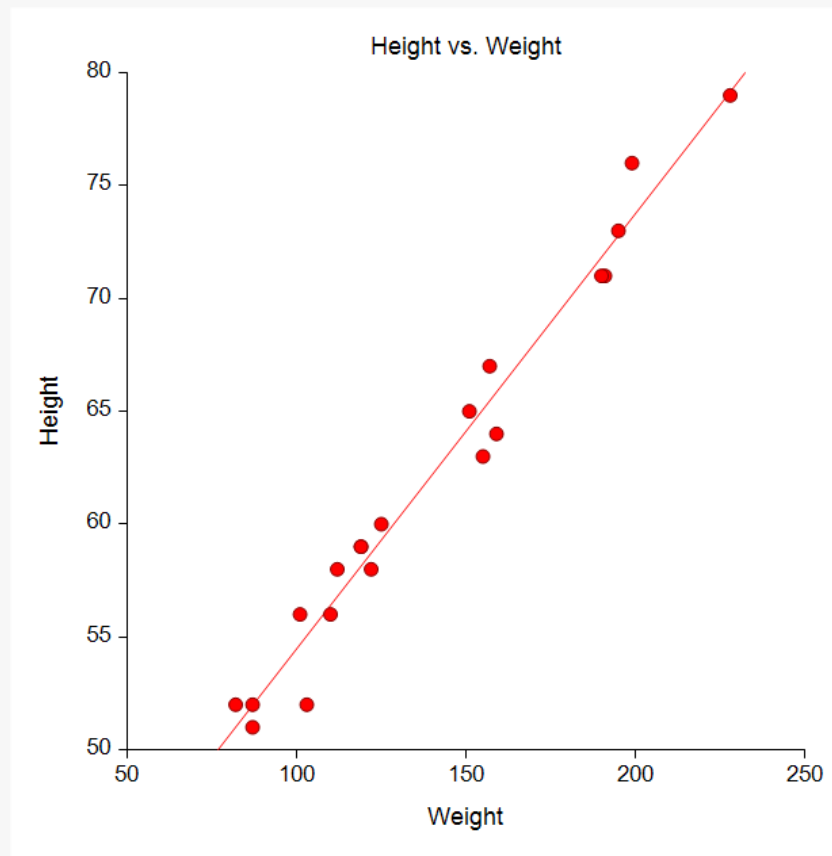
The plot shows the data and the linear regression line. This plot is very useful for finding outliers and nonlinearities. It gives you a good feel for how well the linear regression model fits the data.

Run Summary Section

Run Summary Section

Parameter	Value	Parameter	Value
Dependent Variable	Height	Rows Processed	26
Independent Variable	Weight	Rows Used in Estimation	20
Frequency Variable	None	Rows with X Missing	3
Weight Variable	None	Rows with Freq Missing	0
Intercept	35.1337	Rows Prediction Only	3
Slope	0.1932	Sum of Frequencies	20
R-Squared	0.9738	Sum of Weights	20.0000
Correlation	0.9868	Coefficient of Variation	0.0226
Mean Square Error	1.970176	Square Root of MSE	1.40363

This report summarizes the linear regression results. It presents the variables used, the number of rows used, and the basic least squares results. These values are repeated later in specific reports, so they will not be discussed further here.

Coefficient of Variation

The coefficient of variation is a relative measure of dispersion, computed by dividing the square root of the mean square error by the mean of *Y*. By itself, it has little value, but it can be useful in comparative studies.

$$CV = \frac{\sqrt{MSE}}{\bar{Y}}$$

Summary Statement

Summary Statement
───

The equation of the straight line relating Height and Weight is estimated as: Height = (35.1337) + (0.1932) Weight using the 20 observations in this dataset. The y-intercept, the estimated value of Height when Weight is zero, is 35.1337 with a standard error of 1.0887. The slope, the estimated change in Height per unit change in Weight, is 0.1932 with a standard error of 0.0075. The value of R-Squared, the proportion of the variation in Height that can be accounted for by variation in Weight, is 0.9738. The correlation between Height and Weight is 0.9868.

A significance test that the slope is zero resulted in a t-value of 25.8679. The significance level of this t-test is 0.0000. Since 0.0000 < 0.0500, the hypothesis that the slope is zero is rejected.

The estimated slope is 0.1932. The lower limit of the 95% confidence interval for the slope is 0.1775 and the upper limit is 0.2089. The estimated intercept is 35.1337. The lower limit of the 95% confidence interval for the intercept is 32.8464 and the upper limit is 37.4209.
───

This report explains the results in text format.

Descriptive Statistics Section

Descriptive Statistics Section
───

Parameter	Dependent	Independent
Variable	Height	Weight
Count	20	20
Mean	62.1000	139.6000
Standard Deviation	8.4411	43.1221
Minimum	51.0000	82.0000
Maximum	79.0000	228.0000
───

This report presents the mean, standard deviation, minimum, and maximum of the two variables. It is particularly useful for checking that the correct variables were selected.

Regression Estimation Section

Regression Estimation Section
_____

Parameter	Intercept B(0)	Slope B(1)
Regression Coefficients	35.1337	0.1932
Lower 95% Confidence Limit	32.8464	0.1775
Upper 95% Confidence Limit	37.4209	0.2089
Standard Error	1.0887	0.0075
Standardized Coefficient	0.0000	0.9868
T Value	32.2716	25.8679
Prob Level (T Test)	0.0000	0.0000
Prob Level (Randomization Test*)		0.0010
Reject H0 (Alpha = 0.0500)	Yes	Yes
Power (Alpha = 0.0500)	1.0000	1.0000
Regression of Y on X	35.1337	0.1932
Inverse Regression from X on Y	34.4083	0.1984
Orthogonal Regression of Y and X	35.1076	0.1934

Estimated Model
_____

(35.1336680743148) + (0.193168566802902) * (Weight)

_____

* Number of Monte Carlo Samples = 1000, User-Entered Random Seed = 3118927.

Notes:
The above report shows the least-squares estimates of the intercept and slope followed by the corresponding standard errors, confidence intervals, and hypothesis tests. Note that these results are based on several assumptions that should be validated before they are used.

This section reports the values and significance tests of the regression coefficients. Before using this report, check that the assumptions are reasonable by looking at the tests of assumptions report.

Regression Coefficients

The regression coefficients are the least-squares estimates of the *Y*-intercept and the slope. The slope indicates how much of a change in *Y* occurs for a one-unit change in *X*.

Lower - Upper 95% Confidence Limits

These are the lower and upper values of a 100(1 - *α*)% interval estimate for β_j based on a *t*-distribution with *N* - 2 degrees of freedom. This interval estimate assumes that the residuals for the regression model are normally distributed.

The formulas for the lower and upper confidence limits are

$$b_j \pm t_{1-\alpha/2, n-2} s_{b_j}$$

Standard Error

The standard error of the regression coefficient, s_{b_j}, is the standard deviation of the estimate. It provides a measure of the precision of the estimated regression coefficient. It is used in hypothesis tests or confidence limits.

Standardized Coefficient

Standardized regression coefficients are the coefficients that would be obtained if you standardized both variables. Here *standardizing* is defined as subtracting the mean and dividing by the standard deviation of a variable. A regression analysis on these standardized variables would yield these standardized coefficients.

The formula for the standardized regression coefficient is:

$$b_{1,std} = b_1 \left(\frac{s_X}{s_Y} \right)$$

where s_Y and s_X are the standard deviations for the dependent and independent variables, respectively.

Note that in the case of linear regression, the standardized coefficient is equal to the correlation between the two variables.

T-Value

These are the *t*-test values for testing the hypotheses that the intercept and the slope are zero versus the alternative that they are nonzero. These *t*-values have *N - 2* degrees of freedom.

To test that the slope is equal to a hypothesized value other than zero, inspect the confidence limits. If the hypothesized value is outside the confidence limits, the hypothesis is rejected. Otherwise, it is not rejected.

Prob Level

This is the two-sided *p*-value for the significance test of the regression coefficient. The *p*-value is the probability that this *t*-statistic will take on a value at least as extreme as the actually observed value, assuming that the null hypothesis is true (i.e., the regression estimate is equal to zero). If the *p*-value is less than alpha, say 0.05, the null hypothesis is rejected.

Prob Level (Randomization Test)

This is the two-sided *p*-value for the randomization test of whether the slope is zero. Since this value is based on a randomization test, it does not require all of the assumptions that the *t*-test does. The number of Monte Carlo samples of the permutation distribution of the slope is shown in parentheses.

Reject H0 (Alpha = 0.05)

This value indicates whether the null hypothesis was rejected. Note that the level of significance was specified as the value of *Alpha*.

Power (Alpha = 0.05)

Power is the probability of rejecting the null hypothesis that the regression coefficient is zero when in truth, the regression coefficient is some value other than zero. The power is calculated for the case when the estimate coefficient is the actual coefficient, the estimate variance is the true variance, and Alpha is the given value.

High power is desirable. High power means that there is a high probability of rejecting the null hypothesis when the null hypothesis is false. This is a critical measure of sensitivity in hypothesis testing. This estimate of power is based upon the assumption that the residuals are normally distributed.

Regression of Y on X

These are the usual least squares estimates of the intercept and slope from a linear regression of *Y* on *X*. These quantities were given earlier and are reproduced here to allow easy comparisons.

Regression of X on Y

These are the estimated intercept and slope derived from the coefficients of linear regression of *X* on *Y*. These quantities may be useful in calibration and inverse prediction.

Orthogonal Regression of Y and X

These are the estimates of the intercept and slope from an orthogonal regression of *Y* on *X*. This equation minimizes the sum of the squared perpendicular distances between the points and the regression line.

Estimated Model

This is the least squares regression line presented in double precision. Besides showing the regression model in long form, it may be used as a transformation by copying and pasting it into the Transformation portion of the spreadsheet.

Bootstrap Section

Bootstrap Section

Estimation Results			Bootstrap Confidence Limits		
Parameter	Estimate	|	Conf. Level	Lower	Upper
Intercept					
Original Value	35.1337	|	0.9000	33.4868	36.7965
Bootstrap Mean	35.1470	|	0.9500	33.1653	37.3351
Bias (BM - OV)	0.0133	|	0.9900	32.6453	38.1047
Bias Corrected	35.1204				
Standard Error	1.0327				
Slope					
Original Value	0.1932	|	0.9000	0.1815	0.2049
Bootstrap Mean	0.1931	|	0.9500	0.1782	0.2075
Bias (BM - OV)	-0.0001	|	0.9900	0.1703	0.2108
Bias Corrected	0.1932				
Standard Error	0.0071				
Correlation					
Original Value	0.9868	|	0.9000	0.9797	0.9970
Bootstrap Mean	0.9867	|	0.9500	0.9788	1.0000
Bias (BM - OV)	-0.0001	|	0.9900	0.9766	1.0000
Bias Corrected	0.9869				
Standard Error	0.0054				
R-Squared					
Original Value	0.9738	|	0.9000	0.9597	0.9937
Bootstrap Mean	0.9736	|	0.9500	0.9580	1.0000
Bias (BM - OV)	-0.0002	|	0.9900	0.9535	1.0000
Bias Corrected	0.9740				
Standard Error	0.0105				

Standard Error of Estimate

Original Value	1.4036	|	0.9000	1.1608	1.8326
Bootstrap Mean	1.3204	|	0.9500	1.1275	1.9126
Bias (BM - OV)	-0.0833	|	0.9900	1.0015	2.0661
Bias Corrected	1.4869				
Standard Error	0.2051				

Orthogonal Intercept

Original Value	35.1076	|	0.9000	33.4494	36.7878
Bootstrap Mean	35.1205	|	0.9500	33.1306	37.3240
Bias (BM - OV)	0.0129	|	0.9900	32.6325	38.1117
Bias Corrected	35.0946				
Standard Error	1.0383				

Orthogonal Slope

Original Value	0.1934	|	0.9000	0.1817	0.2051
Bootstrap Mean	0.1933	|	0.9500	0.1785	0.2078
Bias (BM - OV)	-0.0001	|	0.9900	0.1703	0.2111
Bias Corrected	0.1934				
Standard Error	0.0072				

Predicted Mean and Confidence Limits of Height when Weight = 90.0000

Original Value	52.5188	|	0.9000	51.8013	53.3283
Bootstrap Mean	52.5252	|	0.9500	51.6985	53.5164
Bias (BM - OV)	0.0064	|	0.9900	51.4953	53.8233
Bias Corrected	52.5125				
Standard Error	0.4618				

.
.
.

(More estimates follow for each Weight)

_____

Sampling Method = Observation, Confidence Limit Type = Reflection, Number of Samples = 1000,
User-Entered Random Seed = 3118927.

Notes:
The main purpose of this report is to present the bootstrap confidence intervals of various parameters. All gross outliers should
have been removed. The sample size should be at least 50 and the sample should be 'representative' of the population it was
drawn from.

This report provides bootstrap estimates of the slope and intercept of the least squares regression line and the orthogonal regression line, the correlation coefficient, and other linear regression quantities. Note that bootstrap confidence intervals and prediction intervals are provided for each of the *X* (Weight) values. Details of the bootstrap method were presented earlier in this chapter.

Note that since these results are based on 3000 random bootstrap samples, they will differ slightly from the results you obtain when you run this report.

Original Value

This is the parameter estimate obtained from the complete sample without bootstrapping.

Bootstrap Mean

This is the average of the parameter estimates of the bootstrap samples.

Bias (BM - OV)

This is an estimate of the bias in the original estimate. It is computed by subtracting the original value from the bootstrap mean.

Bias Corrected

This is an estimated of the parameter that has been corrected for its bias. The correction is made by subtracting the estimated bias from the original parameter estimate.

Standard Error

This is the bootstrap method's estimate of the standard error of the parameter estimate. It is simply the standard deviation of the parameter estimate computed from the bootstrap estimates.

Conf. Level

This is the confidence coefficient of the bootstrap confidence interval given to the right.

Bootstrap Confidence Limits - Lower and Upper

These are the limits of the bootstrap confidence interval with the confidence coefficient given to the left. These limits are computed using the confidence interval method (percentile or reflection) designated on the Bootstrap panel.

Note that to be accurate, these intervals must be based on over a thousand bootstrap samples and the original sample must be representative of the population.

Bootstrap Histograms Section

Bootstrap Histograms Section
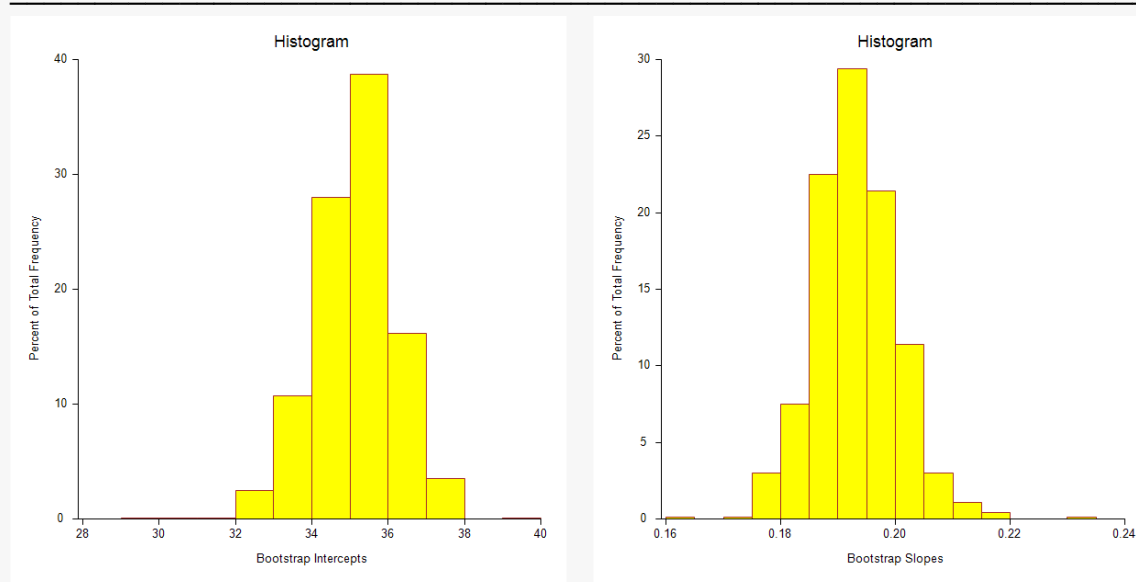
(5 more histograms shown)

Each histogram shows the distribution of the corresponding parameter estimate.

Correlation and R-Squared Section

Correlation and R-Squared Section

Parameter	Pearson Correlation Coefficient	R-Squared	Spearman Rank Correlation Coefficient
Estimated Value	0.9868	0.9738	0.9759
Lower 95% Conf. Limit (r dist'n)	0.9646		
Upper 95% Conf. Limit (r dist'n)	0.9945		
Lower 95% Conf. Limit (Fisher's z)	0.9662		0.9387
Upper 95% Conf. Limit (Fisher's z)	0.9949		0.9906
Adjusted (Rbar)		0.9723	
T-Value for H0: Rho = 0	25.8679	25.8679	18.9539
Prob Level for H0: Rho = 0	0.0000	0.0000	0.0000
Prob Level (Randomization Test N =1000)	0.0010		

Notes:
The confidence interval for the Pearson correlation assumes that X and Y follow the bivariate normal distribution. This is a different assumption from linear regression which assumes that X is fixed and Y is normally distributed.

Two confidence intervals are given. The first is based on the exact distribution of Pearson's correlation. The second is based on Fisher's z transformation which approximates the exact distribution using the normal distribution. Why are both provided? Because most books only mention Fisher's approximate method, it will often be needed to do homework. However, the exact methods should be used whenever possible.

The confidence limits can be used to test hypotheses about the correlation. To test the hypothesis that rho is a specific value, say r0, check to see if r0 is between the confidence limits. If it is, the null hypothesis that rho = r0 is not rejected. If r0 is outside the limits, the null hypothesis is rejected.

Spearman's Rank correlation is calculated by replacing the original data with their ranks. This correlation is used when some of the assumptions may be invalid.

This report provides results about Pearson's correlation, R-squared, and Spearman's rank correlation.

Pearson Correlation Coefficient

Details of the calculation of this value were given earlier in the chapter. Remember that this value is an index of the strength of the linear association between *X* and *Y*. The range of values is from -1 to 1. Strong association occurs when the magnitude of the correlation is close to one. Low correlations are those near zero.

Two sets of confidence limits are given. The first is a set of exact limits computed from the distribution of the correlation coefficient. These limits assume that *X* and *Y* follow the bivariate normal distribution. The second set of limits are limits developed by R. A. Fisher as an approximation to the exact limits. The approximation is quite good as you can see by comparing the two sets of limits. The second set is provided because they are often found in statistics books. In most cases, you should use the first set based on the *r* distribution because they are exact. You may want to compare these limits with those found for the correlation in the Bootstrap report.

The two-sided hypothesis test and probability level are for testing whether the correlation is zero.

Prob Level (Randomization Test)

This is the two-sided *p*-value for the randomization test of whether the slope is zero. This probability value may also be used to test whether the Pearson correlation is zero. Since this value is based on a randomization test, it does not require all of the assumptions that the parametric test does. The number of Monte Carlo samples of the permutation distribution of the slope is shown in parentheses.

Spearman Rank Correlation Coefficient

The Spearman's rank correlation is simply the Pearson correlation computed on the ranks of *X* and *Y* rather than on the actual data. By using the ranks, some of the assumptions may be relaxed. However, the interpretation of the correlation is much more difficult.

The confidence interval for this correlation is calculated using the Fisher's z transformation of the rank correlation.

The two-sided hypothesis test and probability level are for testing whether the rank correlation is zero.

R-Squared

R^2, officially known as the coefficient of determination, is defined as

$$R^2 = \frac{SS_{Model}}{SS_{Total}}$$

R^2 is probably the most popular statistical measure of how well the regression model fits the data. R^2 may be defined either as a ratio or a percentage. Since we use the ratio form, its values range from zero to one. A value of R^2 near zero indicates no linear relationship between the *Y* and *X,* while a value near one indicates a perfect linear fit. Although popular, R^2 should not be used indiscriminately or interpreted without scatter plot support. Following are some qualifications on its interpretation:

1. *Linearity*. R^2 does not measure the appropriateness of a linear model. It measures the strength of the linear component of the model. Suppose the relationship between *X* and *Y* was a perfect circle. The R^2 value of this relationship would be zero.

2. *Predictability*. A large R^2 does not necessarily mean high predictability, nor does a low R^2 necessarily mean poor predictability.

3. *No-intercept model*. The definition of R^2 assumes that there is an intercept in the regression model. When the intercept is left out of the model, the definition of R^2 changes dramatically. The fact that your R^2 R^2 value increases when you remove the intercept from the regression model does not reflect an increase in the goodness of fit. Rather, it reflects a change in the underlying meaning of R^2.

4. *Sample size*. R^2 is highly sensitive to the number of observations. The smaller the sample size, the larger its value.

Adjusted R-Squared

This is an adjusted version of R^2. The adjustment seeks to remove the distortion due to a small sample size.

$$R^2_{adjusted} = 1 - (1 - R^2)\left(\frac{N-1}{N-2}\right)$$

Analysis of Variance Section

Analysis of Variance Section

Source	DF	Sum of Squares	Mean Square	F-Ratio	Prob Level	Power (5%)
Intercept	1	77128.2	77128.2			
Slope	1	1318.337	1318.337	669.1468	0.0000	1.0000
Error	18	35.46317	1.970176			
Lack of Fit	16	34.96317	2.185198	8.7408	0.1074	
Pure Error	2	0.5	0.25			
Adj. Total	19	1353.8	71.25263			
Total	20	78482				

Standard Deviation of Residuals

s = Square Root(1.970176) = 1.40363

Notes:
The above report shows the F-Ratio for testing whether the slope is zero, the degrees of freedom, and the mean square error.
The mean square error, which estimates the variance of the residuals, is used extensively in the calculation of hypothesis tests and confidence intervals.

An analysis of variance (ANOVA) table summarizes the information related to the sources of variation in data.

Source

This represents the partitions of the variation in Y. There are four sources of variation listed: intercept, slope, error, and total (adjusted for the mean).

DF

The degrees of freedom are the number of dimensions associated with this term. Note that each observation can be interpreted as a dimension in N-dimensional space. The degrees of freedom for the intercept, model, error, and adjusted total are 1, *1*, N - 2, and N - 1, respectively.

Sum of Squares

These are the sums of squares associated with the corresponding sources of variation. Note that these values are in terms of the dependent variable, Y. The formulas for each are

$$SS_{intercept} = N\bar{Y}^2$$

$$SS_{slope} = \Sigma(\hat{Y} - \bar{Y})^2$$

$$SS_{error} = \Sigma(Y - \hat{Y})^2$$

$$SS_{total} = \Sigma(Y - \bar{Y})^2$$

Note that the *lack of fit* and *pure error* values are provided if there are observations with identical values of the independent variable.

Mean Square

The mean square is the sum of squares divided by the degrees of freedom. This mean square is an estimated variance. For example, the mean square error is the estimated variance of the residuals (the residuals are sometimes called the *errors*).

F-Ratio

This is the *F* statistic for testing the null hypothesis that the slope equals zero. This *F*-statistic has 1 degree of freedom for the numerator variance and *N* - 2 degrees of freedom for the denominator variance.

Prob Level

This is the *p*-value for the above *F* test. The *p*-value is the probability that the test statistic will take on a value at least as extreme as the observed value, assuming that the null hypothesis is true. If the *p*-value is less than alpha, say 0.05, the null hypothesis is rejected. If the *p*-value is greater than alpha, the null hypothesis is accepted.

Power(5%)

Power is the probability of rejecting the null hypothesis that the slope is zero when it is not.

S = Root Mean Square Error

s is the square root of the mean square error. It is an estimate of the standard deviation of the residuals.

Summary Matrices

Summary Matrices

Index	X'X 0	X'X 1	X'Y 2	X'X Inverse 0	X'X Inverse 1
0	20	2792	1242	0.6015912	-0.003951227
1	2792	425094	180208	-0.003951227	2.830392E-05
2 (Y'Y)			78482		
Determinant		706616			1.415196E-06

Variance - Covariance Matrix of Regression Coefficients

Index	VC(b) 0	VC(b) 1
0	1.185241	-0.007784612
1	-0.007784612	5.576369E-05

This section provides the matrices from which the least square regression values are calculated. Occasionally, these values may be useful in hand calculations.

Tests of Assumptions Section

Tests of Assumptions Section
_____

Assumption/Test	Test Value	Prob Level	Is the Assumption Reasonable at the 0.2000 Level of Significance?
Residuals follow Normal Distribution?			
Shapiro Wilk	0.9728	0.812919	Yes
Anderson Darling	0.2751	0.660885	Yes
D'Agostino Skewness	-0.9590	0.337543	Yes
D'Agostino Kurtosis	0.1205	0.904066	Yes
D'Agostino Omnibus	0.9343	0.626796	Yes
Constant Residual Variance?			
Modified Levene Test	0.0946	0.761964	Yes
Relationship is a Straight Line?			
Lack of Linear Fit F(16, 2) Test	8.7408	0.107381	No
No Serial Correlation?			

Evaluate the Serial-Correlation report and the Durbin-Watson test if you have equal-spaced time series data.
_____

Notes:
A "Yes" means there is not enough evidence to make this assumption seem unreasonable. This lack of evidence may be because the sample size is too small, the assumptions of the test itself are not met, or the assumption is valid.
A "No" means the that the assumption is not reasonable. However, since these tests are related to sample size, you should assess the role of sample size in the tests by also evaluating the appropriate plots and graphs. A large dataset (say N > 500) will often fail at least one of the normality tests because it is hard to find a large dataset that is perfectly normal.

Normality and Constant Residual Variance:
Possible remedies for the failure of these assumptions include using a transformation of Y such as the log or square root, correcting data-recording errors found by looking into outliers, adding additional independent variables, using robust regression, or using bootstrap methods.

Straight-Line:
Possible remedies for the failure of this assumption include using nonlinear regression or polynomial regression.

This report presents numeric tests of some of the assumptions made when using linear regression. The results of these tests should be compared to an appropriate plot to determine if the assumption is valid or not.

Note that a 'Yes' means that there is not enough evidence to reject the assumption. This lack of assumption test rejection may be because the sample size is too small, or the assumptions of the test were no met. It does not necessarily mean that the data met assumption. Likewise, a 'No' may occur because the sample size is very large. It is almost always possible to fail a preliminary test given a large enough sample size. No assumption is every fits perfectly. Bottom line, you should also investigate plots designed to check the assumptions.

Residuals follow Normal Distribution?

This section displays the results of five normality tests of the residuals. The Shapiro-Wilk and Anderson-Darling tests are usually considered as the best.

Unfortunately, these tests have small statistical power (probability of detecting nonnormal data) unless the sample sizes are large, say over 300. Hence, if the decision is to reject normality, you can be reasonably certain that the data are not normal. However, if the decision is not to reject, the situation is not as clear. If you have a sample size of 300 or more, you can reasonably assume that the actual distribution is closely approximated by the normal distribution. If your sample size is less than 300, all you know for sure is that

there was not enough evidence in your data to reject the normality of residuals assumption. In other words, the data might be nonnormal, you just could not prove it. In this case, you must rely on the graphics to justify the normality assumption.

Shapiro-Wilk W Test

This test for normality, developed by Shapiro and Wilk (1965), has been found to be the most powerful test in most situations. It is the ratio of two estimates of the variance of a normal distribution based on a random sample of N observations. The numerator is proportional to the square of the best linear estimator of the standard deviation. The denominator is the sum of squares of the observations about the sample mean. W may be written as the square of the Pearson correlation coefficient between the ordered observations and a set of weights which are used to calculate the numerator. Since these weights are asymptotically proportional to the corresponding expected normal order statistics, W is roughly a measure of the straightness of the normal quantile-quantile plot. Hence, the closer W is to one, the more normal the sample is.

The probability values for W are valid for samples in the range of 3 to 5000.

The test is not calculated when a frequency variable is specified.

Anderson-Darling Test

This test, developed by Anderson and Darling (1954), is based on EDF statistics. In some situations, it has been found to be as powerful as the Shapiro-Wilk test.

The test is not calculated when a frequency variable is specified.

D'Agostino Skewness

D'Agostino (1990) proposed a normality test based on the skewness coefficient, $\sqrt{b_1}$. Because the normal distribution is symmetrical, $\sqrt{b_1}$ is equal to zero for normal data. Hence, a test can be developed to determine if the value of $\sqrt{b_1}$ is significantly different from zero. If it is, the data are obviously nonnormal. The test statistic is, under the null hypothesis of normality, approximately normally distributed. The computation of this statistic is restricted to sample sizes greater than 8. The formula and further details are given in the Descriptive Statistics chapter.

D'Agostino Kurtosis

D'Agostino (1990) proposed a normality test based on the kurtosis coefficient, b_2. For the normal distribution, the theoretical value of b_2 is 3. Hence, a test can be developed to determine if the value of b_2 is significantly different from 3. If it is, the residuals are obviously nonnormal. The test statistic is, under the null hypothesis of normality, approximately normally distributed for sample sizes $N > 20$. The formula and further details are given in the Descriptive Statistics chapter.

D'Agostino Omnibus

D'Agostino (1990) proposed a normality test that combines the tests for skewness and kurtosis. The statistic, K^2, is approximately distributed as a chi-square with two degrees of freedom.

Constant Residual Variance?

Linear regression assumes that the residuals have constant variance. The validity of this assumption can be checked by looking at a plot of the absolute values of the residuals versus the *X* variable. The modified Levene test may be used when a numerical answer is needed.

If your data fail this test, you may want to use a logarithm transformation or a weighted regression.

Modified Levene Test

The *modified Levene test* can be used to evaluate the validity of the assumption of constant variance. It has been shown to be reliable even when the residuals do not follow a normal distribution. The mathematical details of the test were presented earlier in this chapter.

Relationship is a Straight Line?

Linear regression assumes that the relationship between *X* and *Y* is a straight line (linear). The validity of this assumption can be checked by looking at the plot *Y* versus *X* and at the plot of the residuals versus *X*. The lack of fit test may be used when a numerical answer is needed.

If your data fail this test, you may want to use a different model which accounts for the curvature. The Growth and Other Models procedure in curve fitting is a good choice when curvature exists in your data.

Lack of Linear Fit Test

The *lack-of-fit* test is used to test for a departure from the linear fit. This test requires that there are multiple observations for at least one *X* value. When such is the case, an estimate of *pure error* and *lack of fit* can be found, and an *F* test created. The mathematical details of the test were presented earlier in this chapter.

Serial Correlation and Durbin-Watson Sections

Serial Correlation of Residuals Section

Lag	Serial Correlation	Lag	Serial Correlation	Lag	Serial Correlation
1	0.1029	9	-0.2353	17	
2	-0.4127*	10	-0.0827	18	
3	0.0340	11	-0.0316	19	
4	0.2171	12	-0.0481	20	
5	-0.1968	13	0.0744	21	
6	-0.0194	14	0.0073	22	
7	0.2531	15		23	
8	-0.0744	16		24	

Notes:
Each serial correlation is the Pearson correlation calculated between the original series of residuals and the residuals lagged the specified number of periods. This feature of residuals is only meaningful for data obtained sorted in time order. One of the assumptions is that none of these serial correlations is significant. Starred correlations are those for which |Fisher's Z| > 1.645 which indicates whether the serial correlation is 'large.'

If serial correlation is detected in time series data, the remedy is to account for it either by replacing Y with first differences or by fitting the serial pattern using a method such as that proposed by Cochrane and Orcutt.

Durbin-Watson Test For Serial Correlation
―――

Parameter	Value	Did the Test Reject H0: Rho(1) = 0?
Durbin-Watson Value	1.6978	
Prob. Level: Positive Serial Correlation	0.2366	No
Prob. Level: Negative Serial Correlation	0.7460	No
―――

Notes:
The Durbin-Watson test was created to test for first-order serial correlation in regression data taken over time. If the rows of your dataset do not represent successive time periods, you should ignore this test.

This report gives the probability of rejecting the null hypothesis of no first-order serial correlation. Possible remedies for serial correlation were given in the Notes to the Serial Correlation report, above.

This section reports on the autocorrelation structure of the residuals. Of course, if your data were not taken through time, this section should be ignored.

Lag

The lag, *k,* is the number of periods back.

Serial Correlation

The serial correlation reported here is the sample autocorrelation coefficient of lag *k*. It is computed as

$$r_k = \frac{\sum e_{i-k}e_i}{\sum e_i^2} \quad \text{for } k = 1, 2, \ldots, 24$$

The distribution of these autocorrelations may be approximated by the distribution of the regular correlation coefficient. Using this fact, Fisher's *Z* transformation may be used to find large autocorrelations. If the Fisher's *Z* transformation of the autocorrelation is greater than 1.645, the autocorrelation is assumed to be large and the observation is starred.

Durbin-Watson Value

The Durbin-Watson test is often used to test for positive or negative, first-order, serial correlation. It is calculated as follows

$$DW = \frac{\sum_{j=2}^{N}\left(e_j - e_{j-1}\right)^2}{\sum_{j=1}^{N} e_j^2}$$

The distribution of this test is mathematically difficult because it involves the *X* values. Originally, Durbin-Watson (1950, 1951) gave a pair of bounds to be used. However, there is a large range of indecision that can be found when using these bounds. Instead of using these bounds, **NCSS** calculates the exact probability using the beta distribution approximation suggested by Durbin-Watson (1951). This approximation has been shown to be accurate to three decimal places in most cases.

PRESS Section

PRESS Section

Parameter	From PRESS Residuals	From Regular Residuals
Sum of Squared Residuals	43.15799	35.46317
Sum of |Residuals|	24.27421	22.02947
R-Squared	0.9681	0.9738

Notes:
A PRESS residual is found by estimating the regression equation without the observation, predicting the dependent variable, and subtracting the predicted value from the actual value. The PRESS values are calculated from these PRESS residuals. The Regular values are the corresponding calculations based on the regular residuals.

The PRESS values are often used to compare models in a multiple-regression variable selection. They show how well the model predicts observations that were not used in the estimation.

This section reports on the PRESS statistics. The regular statistics, computed on all of the data, are provided to the side to make comparison between corresponding values easier.

Sum of Squared Residuals

PRESS is an acronym for prediction sum of squares. It was developed for use in variable selection to validate a regression model. To calculate PRESS, each observation is individually omitted. The remaining N - 1 observations are used to calculate a regression and estimate the value of the omitted observation. This is done N times, once for each observation. The difference between the actual Y value and the predicted Y with the observation deleted is called the prediction error or PRESS residual. The sum of the squared prediction errors is the PRESS value. The smaller PRESS is, the better the predictability of the model.

Sum of |Press Residuals|

This is the sum of the absolute value of the PRESS residuals or prediction errors. If a large value for the *PRESS* is due to one or a few large PRESS residuals, this statistic may be a more accurate way to evaluate predictability.

Press R-Squared

The PRESS value above can be used to compute an R^2-like statistic, called *R2Predict*, which reflects the prediction ability of the model. This is a good way to validate the prediction of a regression model without selecting another sample or splitting your data. It is very possible to have a high R^2 and a very low *R2Predict*. When this occurs, it implies that the fitted model is data dependent. This *R2Predict* ranges from below zero to above one. When outside the range of zero to one, it is truncated to stay within this range.

Predicted Values and Confidence Limits Section

Predicted Values and Confidence Limits Section

Weight (X)	Predicted Height (Yhat|X)	Standard Error of Yhat	Lower 95% Confidence Limit of Y|X	Upper 95% Confidence Limit of Y|X
90.0000	52.5188	0.4855	51.4989	53.5388
100.0000	54.4505	0.4312	53.5446	55.3565
150.0000	64.1090	0.3233	63.4297	64.7882
200.0000	73.7674	0.5495	72.6129	74.9218
250.0000	83.4258	0.8821	81.5725	85.2791

The confidence interval estimates the mean of the Y values in a large sample of individuals with this value of X. The interval is only accurate if all of the linear regression assumptions are valid.

The predicted values and confidence intervals of the mean response of *Y* given *X* are provided here. The values of *X* used here were specified in the *Predict Y at these X Values* option on the *Variables* panel.

It is important to note that violations of any regression assumptions will invalidate this interval estimate.

X

This is the value of *X* at which the prediction is made.

Predicted Y (Yhat|X)

The predicted value of *Y* for the value of *X* indicated.

Standard Error of Yhat

This is the estimated standard deviation of the predicted value.

Lower 95% Confidence Limit of Y|X

This is the lower limit of a 95% confidence interval estimate of the mean of *Y* at this value of *X*.

Upper 95% Confidence Limit of Y|X

This is the upper limit of a 95% confidence interval estimate of the mean of *Y* at this value of *X*. Note that you set the alpha level on the *Variables* panel.

Predicted Values and Prediction Limits Section

Predicted Values and Prediction Limits Section

Weight (X)	Predicted Height (Yhat|X)	Standard Error of Yhat	Lower 95% Prediction Limit of Y|X	Upper 95% Prediction Limit of Y|X
90.0000	52.5188	1.4852	49.3985	55.6392
100.0000	54.4505	1.4684	51.3656	57.5355
150.0000	64.1090	1.4404	61.0828	67.1351
200.0000	73.7674	1.5074	70.6005	76.9342
250.0000	83.4258	1.6578	79.9429	86.9087

The prediction interval estimates the predicted value of Y for a single individual with this value of X. The interval is only accurate if all of the linear regression assumptions are valid.

The predicted values and prediction intervals of the response of *Y* given *X* are provided here. The values of *X* used here were specified in the *Predict Y at these X Values* option on the *Variables* panel.

It is important to note that violations of any regression assumptions will invalidate this interval estimate.

X

This is the value of *X* at which the prediction is made.

Predicted Y (Yhat|X)

The predicted value of *Y* for the value of *X* indicated.

Standard Error of Yhat

This is the estimated standard deviation of the predicted value.

Lower 95% Prediction Limit of Y|X

This is the lower limit of a 95% prediction interval estimate of the mean of *Y* at this value of *X*.

Upper 95% Prediction Limit of Y|X

This is the upper limit of a 95% prediction interval estimate of the mean of *Y* at this value of *X*. Note that you set the alpha level on the *Variables* panel.

Residual Plots

The residuals can be graphically analyzed in numerous ways. For certain, the regression analyst should examine all of the basic residual graphs: the histogram, the density trace, the normal probability plot, the serial correlation plots (for time series data), the scatter plot of the residuals versus the sequence of the observations (for time series data), and the scatter plot of the residuals versus the independent variable.

For the scatter plots of residuals versus either the predicted values of *Y* or the independent variables, Hoaglin (1983) explains that there are several patterns to look for. You should note that these patterns are very difficult, if not impossible, to recognize for small data sets.

Point Cloud

A point cloud, basically in the shape of a rectangle or a horizontal band, would indicate no relationship between the residuals and the variable plotted against them. This is the preferred condition.

Wedge

An increasing or decreasing wedge would be evidence that there is increasing or decreasing (nonconstant) variation. A transformation of *Y* may correct the problem or weighted least squares may be needed.

Bowtie

This is similar to the wedge above in that the residual plot shows a decreasing wedge in one direction while simultaneously having an increasing wedge in the other direction. A transformation of *Y* may correct the problem or weighted least squares may be needed.

Sloping Band

This kind of residual plot suggests adding a linear version of the independent variable to the model.

Curved Band

This kind of residual plot may be indicative of a nonlinear relationship between *Y* and the independent variable that was not accounted for. The solution might be to use a transformation on *Y* to create a linear relationship with *X*. Another possibility might be to add quadratic or cubic terms of a particular independent variable.

Curved Band with Increasing or Decreasing Variability

This residual plot is really a combination of the wedge and the curved band. It too must be avoided.

Residuals Plots

Residuals vs X Plot

Residuals of Height vs. Weight
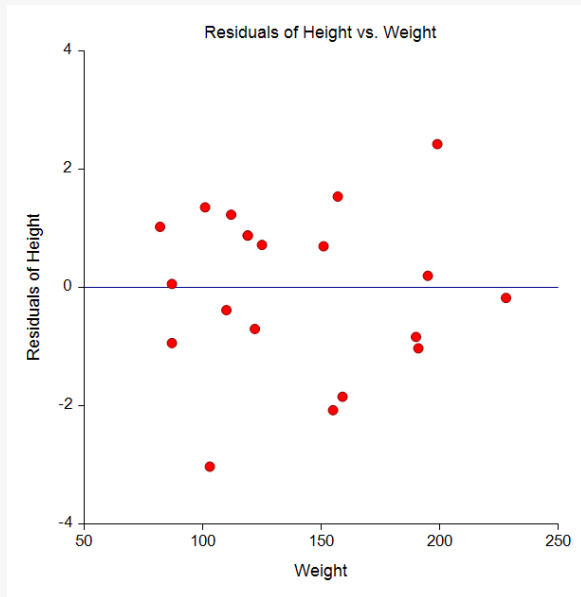
This plot is useful for showing nonlinear patterns and outliers. The preferred pattern is a rectangular shape or point cloud. Any other nonrandom pattern may require a redefining of the regression model.

|Residuals| vs X Plot

|Residuals(Height)| vs. Weight
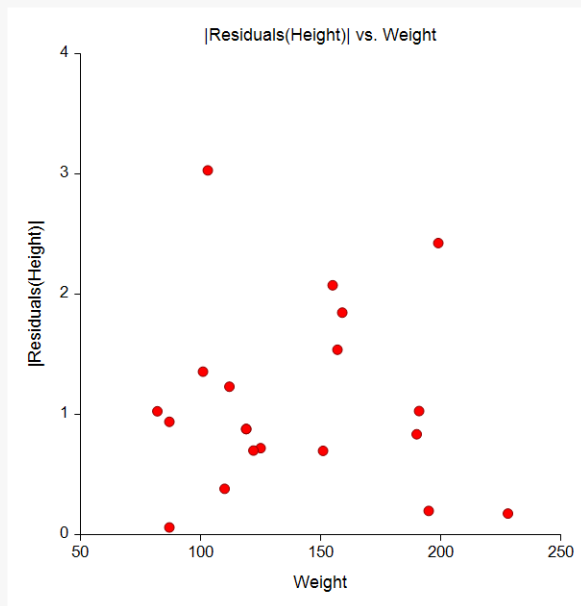
This plot is useful for showing nonconstant variance in the residuals. The preferred pattern is a rectangular shape or point cloud. The most common type of nonconstant variance occurs when the variance is proportion to X. This is shown by a funnel shape. Remedies for nonconstant variances were discussed earlier.

RStudent vs X Plot
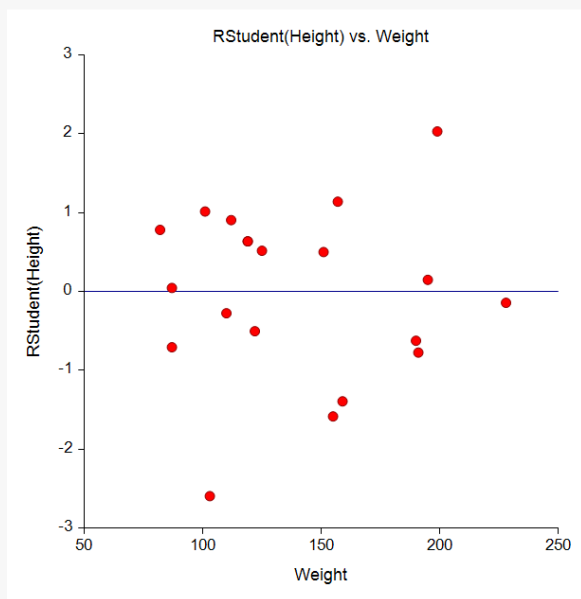
This is a scatter plot of the RStudent residuals versus the independent variable. The preferred pattern is a rectangular shape or point cloud. This plot is helpful in identifying any outliers.

Sequence Plot: Residuals vs Row
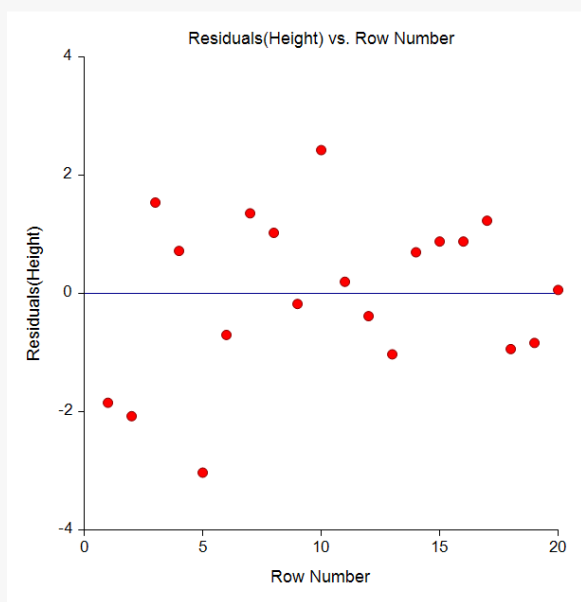
Sequence plots may be useful in finding variables that are not accounted for by the regression equation. They are especially useful if the data were taken over time.

Serial Correlation Plot: Residuals vs Lagged Residuals
_____
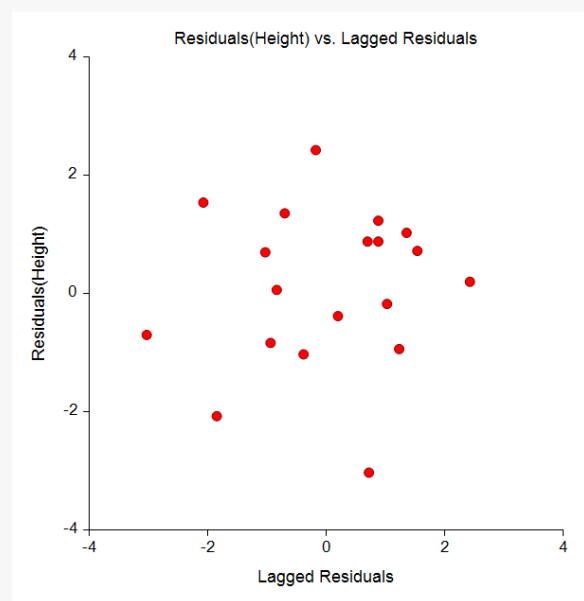
This is a scatter plot of the ith residual versus the ith-1 residual. It is only useful for time series data where the order of the rows on the database is important.

The purpose of this plot is to check for first-order autocorrelation. You would like to see a random pattern, i.e., a rectangular or uniform distribution of the points. A strong positive or negative trend indicates a need to redefine the model with some type of autocorrelation component.

Positive autocorrelation or serial correlation means that the residual in time period t tends to have the same sign as the residual in time period (t - 1). On the other hand, a strong negative autocorrelation means that the residual in time period t tends to have the opposite sign as the residual in time period (t - 1).

Be sure to check the Durbin-Watson statistic.

Distributional Plots of Residuals

Histogram
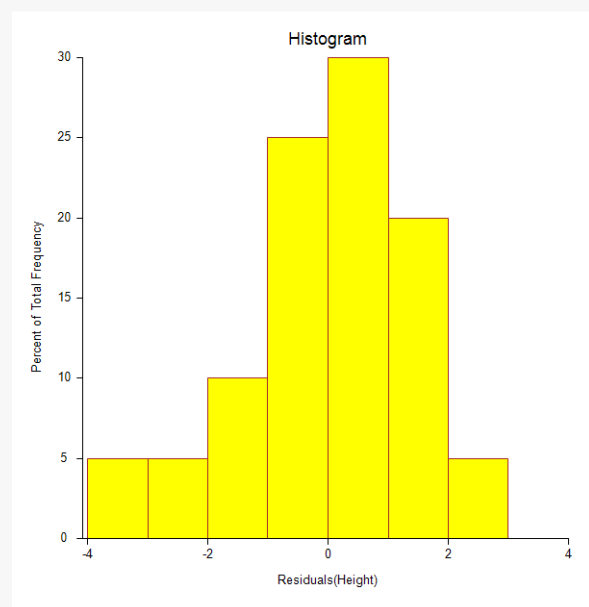
The purpose of the histogram and density trace of the residuals is to evaluate whether they are normally distributed. Unless you have a large sample size, it is best not to rely on the histogram for visually evaluating normality of the residuals. The better choice would be the normal probability plot.

Probability Plot
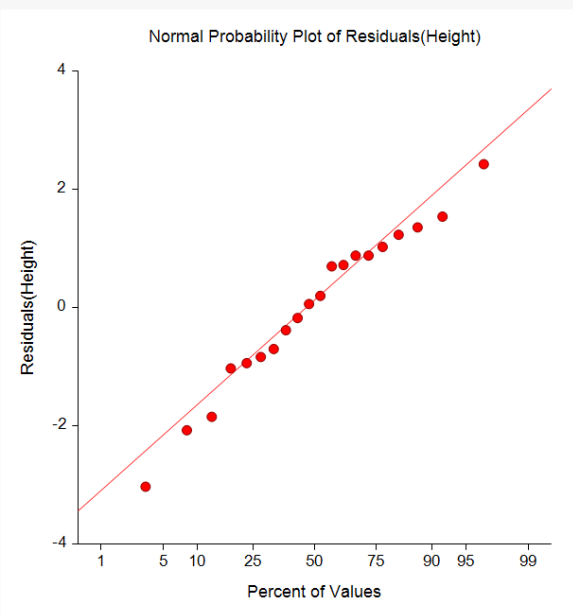
If the residuals are normally distributed, the data points of the normal probability plot will fall along a straight line. Major deviations from this ideal picture reflect departures from normality. Stragglers at either end of the normal probability plot indicate outliers. Curvature at both ends of the plot indicates long or

short distributional tails. Convex, or concave, curvature indicates a lack of symmetry. Gaps, plateaus, or segmentation indicate clustering and may require a closer examination of the data or model. Of course, use of this graphic tool with very small sample sizes is unwise.

If the residuals are not normally distributed, the t-tests on regression coefficients, the F-tests, and the interval estimates are not valid. This is a critical assumption to check.

Original Data Section

Original Data Section

Row	Weight (X)	Height (Y)	Predicted Height (Yhat|X)	Residual
1	159.0000	64.0000	65.8475	-1.8475
2	155.0000	63.0000	65.0748	-2.0748
3	157.0000	67.0000	65.4611	1.5389
4	125.0000	60.0000	59.2797	0.7203
5	103.0000	52.0000	55.0300	-3.0300
6	122.0000	58.0000	58.7002	-0.7002
7	101.0000	56.0000	54.6437	1.3563
8	82.0000	52.0000	50.9735	1.0265
9	228.0000	79.0000	79.1761	-0.1761
10	199.0000	76.0000	73.5742	2.4258
11	195.0000	73.0000	72.8015	0.1985
12	110.0000	56.0000	56.3822	-0.3822
13	191.0000	71.0000	72.0289	-1.0289
14	151.0000	65.0000	64.3021	0.6979
15	119.0000	59.0000	58.1207	0.8793
16	119.0000	59.0000	58.1207	0.8793
17	112.0000	58.0000	56.7685	1.2315
18	87.0000	51.0000	51.9393	-0.9393
19	190.0000	71.0000	71.8357	-0.8357
20	87.0000	52.0000	51.9393	0.0607
21	100.0000		54.4505	
22	150.0000		64.1090	
23	200.0000		73.7674	
24		50.0000		
25		60.0000		
26		70.0000		

This report provides a data list that may be used to verify whether the correct variables were selected.

This report lists the values of *X*, *Y*, the predicted value of *Y*, and the residual.

Predicted Values and Confidence Limits of Means Section

Predicted Values and Confidence Limits of Means

Row	Weight (X)	Height (Y)	Predicted Height (Yhat|X)	Standard Error of Yhat	Lower 95% Conf. Limit of Y Mean|X	Upper 95% Conf. Limit of Y Mean|X
1	159.0000	64.0000	65.8475	0.3457	65.1212	66.5737
2	155.0000	63.0000	65.0748	0.3343	64.3725	65.7771
3	157.0000	67.0000	65.4611	0.3397	64.7475	66.1748
4	125.0000	60.0000	59.2797	0.3323	58.5817	59.9778
5	103.0000	52.0000	55.0300	0.4162	54.1557	55.9044
6	122.0000	58.0000	58.7002	0.3403	57.9854	59.4151
7	101.0000	56.0000	54.6437	0.4261	53.7484	55.5390
8	82.0000	52.0000	50.9735	0.5325	49.8548	52.0922
9	228.0000	79.0000	79.1761	0.7309	77.6404	80.7118
10	199.0000	76.0000	73.5742	0.5434	72.4326	74.7158
11	195.0000	73.0000	72.8015	0.5193	71.7106	73.8925
12	110.0000	56.0000	56.3822	0.3839	55.5757	57.1887
13	191.0000	71.0000	72.0289	0.4958	70.9872	73.0705
14	151.0000	65.0000	64.3021	0.3252	63.6189	64.9853
15	119.0000	59.0000	58.1207	0.3495	57.3864	58.8551
16	119.0000	59.0000	58.1207	0.3495	57.3864	58.8551
17	112.0000	58.0000	56.7685	0.3755	55.9797	57.5574
18	87.0000	51.0000	51.9393	0.5028	50.8830	52.9956
19	190.0000	71.0000	71.8357	0.4901	70.8061	72.8653
20	87.0000	52.0000	51.9393	0.5028	50.8830	52.9956
21	100.0000		54.4505	0.4312	53.5446	55.3565
22	150.0000		64.1090	0.3233	63.4297	64.7882
23	200.0000		73.7674	0.5495	72.6129	74.9218
24		50.0000				
25		60.0000				
26		70.0000				

The confidence interval estimates the mean of the Y values in a large sample of individuals with this value of X. The interval is only accurate if all of the linear regression assumptions are valid.

The predicted values and confidence intervals of the mean response of *Y* given *X* are given for each observation.

X

This is the value of *X* at which the prediction is made.

Y

This is the actual value of *Y*.

Predicted Y (Yhat|X)

The predicted value of *Y* for the value of *X* indicated.

Standard Error of Yhat

This is the estimated standard deviation of the predicted mean value.

Lower 95% Confidence Limit of Y|X

This is the lower limit of a 95% confidence interval estimate of the mean of *Y* at this value of *X*.

Upper 95% Confidence Limit of Y|X

This is the upper limit of a 95% confidence interval estimate of the mean of *Y* at this value of *X*. Note that you set the alpha level on the *Variables* panel.

Predicted Values and Prediction Limits Section

Predicted Values and Prediction Limits

Row	Weight (X)	Height (Y)	Predicted Height (Yhat|X)	Standard Error of Yhat	Lower 95% Prediction Limit of Y|X	Upper 95% Prediction Limit of Y|X
1	159.0000	64.0000	65.8475	1.4456	62.8104	68.8845
2	155.0000	63.0000	65.0748	1.4429	62.0434	68.1062
3	157.0000	67.0000	65.4611	1.4441	62.4271	68.4952
4	125.0000	60.0000	59.2797	1.4424	56.2493	62.3101
5	103.0000	52.0000	55.0300	1.4640	51.9542	58.1058
6	122.0000	58.0000	58.7002	1.4443	55.6659	61.7346
7	101.0000	56.0000	54.6437	1.4669	51.5619	57.7255
8	82.0000	52.0000	50.9735	1.5012	47.8195	54.1275
9	228.0000	79.0000	79.1761	1.5825	75.8513	82.5009
10	199.0000	76.0000	73.5742	1.5051	70.4120	76.7364
11	195.0000	73.0000	72.8015	1.4966	69.6573	75.9458
12	110.0000	56.0000	56.3822	1.4552	53.3250	59.4394
13	191.0000	71.0000	72.0289	1.4886	68.9014	75.1564
14	151.0000	65.0000	64.3021	1.4408	61.2751	67.3291
15	119.0000	59.0000	58.1207	1.4465	55.0818	61.1597
16	119.0000	59.0000	58.1207	1.4465	55.0818	61.1597
17	112.0000	58.0000	56.7685	1.4530	53.7159	59.8212
18	87.0000	51.0000	51.9393	1.4910	48.8069	55.0717
19	190.0000	71.0000	71.8357	1.4867	68.7122	74.9592
20	87.0000	52.0000	51.9393	1.4910	48.8069	55.0717
21	100.0000		54.4505	1.4684	51.3656	57.5355
22	150.0000		64.1090	1.4404	61.0828	67.1351
23	200.0000		73.7674	1.5074	70.6005	76.9342
24		50.0000				
25		60.0000				
26		70.0000				

The prediction interval estimates the predicted value of Y for a single individual with this value of X. The interval is only accurate if all of the linear regression assumptions are valid.

The predicted values and confidence intervals of the mean response of *Y* given *X* are given for each observation.

X

This is the value of *X* at which the prediction is made.

Y

This is the actual value of *Y*.

Predicted Y (Yhat|X)

The predicted value of *Y* for the value of *X* indicated.

Standard Error of Yhat

This is the estimated standard deviation of the predicted value suitable for creating a prediction limit for an individual.

Lower 95% Prediction Limit of Y|X

This is the lower limit of a 95% prediction interval estimate of *Y* at this value of *X*.

Upper 95% Prediction Limit of Y|X

This is the upper limit of a 95% prediction interval estimate of *Y* at this value of *X*. Note that you set the alpha level on the *Variables* panel.

Working-Hotelling Simultaneous Confidence Band

Working-Hotelling Simultaneous Confidence Band

Row	Weight (X)	Height (Y)	Predicted Height (Yhat|X)	Standard Error of Yhat	Lower 95% Conf. Band of Y Mean|X	Upper 95% Conf. Band of Y Mean|X
1	159.0000	64.0000	65.8475	0.3457	64.8036	66.8914
2	155.0000	63.0000	65.0748	0.3343	64.0654	66.0842
3	157.0000	67.0000	65.4611	0.3397	64.4353	66.4869
4	125.0000	60.0000	59.2797	0.3323	58.2764	60.2831
5	103.0000	52.0000	55.0300	0.4162	53.7732	56.2868
6	122.0000	58.0000	58.7002	0.3403	57.6727	59.7278
7	101.0000	56.0000	54.6437	0.4261	53.3568	55.9306
8	82.0000	52.0000	50.9735	0.5325	49.3655	52.5814
9	228.0000	79.0000	79.1761	0.7309	76.9688	81.3834
10	199.0000	76.0000	73.5742	0.5434	71.9333	75.2151
11	195.0000	73.0000	72.8015	0.5193	71.2334	74.3697
12	110.0000	56.0000	56.3822	0.3839	55.2229	57.5415
13	191.0000	71.0000	72.0289	0.4958	70.5316	73.5261
14	151.0000	65.0000	64.3021	0.3252	63.3201	65.2842
15	119.0000	59.0000	58.1207	0.3495	57.0652	59.1763
16	119.0000	59.0000	58.1207	0.3495	57.0652	59.1763
17	112.0000	58.0000	56.7685	0.3755	55.6347	57.9024
18	87.0000	51.0000	51.9393	0.5028	50.4210	53.4577
19	190.0000	71.0000	71.8357	0.4901	70.3558	73.3156
20	87.0000	52.0000	51.9393	0.5028	50.4210	53.4577
21	100.0000		54.4505	0.4312	53.1483	55.7527
22	150.0000		64.1090	0.3233	63.1326	65.0853
23	200.0000		73.7674	0.5495	72.1080	75.4268
24		50.0000				
25		60.0000				
26		70.0000				

This is a confidence band for the regression line for all possible values of X from -infinity to + infinity. The confidence coefficient is the proportion of time that this procedure yields a band the includes the true regression line when a large number of samples are taken using the same X values as in this sample.

The predicted values and confidence band of the mean response function are given for each observation. Note that this is a confidence band for all possible values of *X* along the real number line. The confidence coefficient is the proportion of time that this procedure yields a band that includes the true regression line when a large number of samples are taken using the *X* values as in this sample.

X

This is the value of *X* at which the prediction is made.

Y

This is the actual value of *Y*.

Predicted Y (Yhat|X)

The predicted value of *Y* for the value of *X* indicated.

Standard Error of Yhat

This is the estimated standard deviation of the predicted mean value.

Lower 95% Confidence Band of Y|X

This is the lower limit of the 95% confidence band for the value of *Y* at this *X*.

Upper 95% Confidence Band of Y|X

This is the upper limit of the 95% confidence band for the value of *Y* at this *X*.

Residual Section

Residual Section

Row	Weight (X)	Height (Y)	Predicted Height (Yhat|X)	Residual	Standardized Residual	Percent Absolute Error
1	159.0000	64.0000	65.8475	-1.8475	-1.3580	2.8867
2	155.0000	63.0000	65.0748	-2.0748	-1.5220	3.2933
3	157.0000	67.0000	65.4611	1.5389	1.1299	2.2968
4	125.0000	60.0000	59.2797	0.7203	0.5282	1.2004
5	103.0000	52.0000	55.0300	-3.0300	-2.2604	5.8270
6	122.0000	58.0000	58.7002	-0.7002	-0.5142	1.2073
7	101.0000	56.0000	54.6437	1.3563	1.0142	2.4220
8	82.0000	52.0000	50.9735	1.0265	0.7904	1.9741
9	228.0000	79.0000	79.1761	-0.1761	-0.1470	0.2229
10	199.0000	76.0000	73.5742	2.4258	1.8744	3.1918
11	195.0000	73.0000	72.8015	0.1985	0.1522	0.2719
12	110.0000	56.0000	56.3822	-0.3822	-0.2831	0.6825
13	191.0000	71.0000	72.0289	-1.0289	-0.7835	1.4491
14	151.0000	65.0000	64.3021	0.6979	0.5111	1.0737
15	119.0000	59.0000	58.1207	0.8793	0.6468	1.4903
16	119.0000	59.0000	58.1207	0.8793	0.6468	1.4903
17	112.0000	58.0000	56.7685	1.2315	0.9105	2.1232
18	87.0000	51.0000	51.9393	-0.9393	-0.7168	1.8418

19	190.0000	71.0000	71.8357	-0.8357	-0.6354	1.1770
20	87.0000	52.0000	51.9393	0.0607	0.0463	0.1167
21	100.0000		54.4505			
22	150.0000		64.1090			
23	200.0000		73.7674			
24		50.0000				
25		60.0000				
26		70.0000				

The residual is the difference between the actual and the predicted Y values. The formula is Residual = Y - Yhat. The Percent Absolute Error is the 100 |Residual| / Y.

This is a report showing the value of the residual at each observation.

X

This is the value of *X* at which the prediction is made.

Y

This is the actual value of *Y*.

Predicted Y (Yhat|X)

The predicted value of *Y* for the value of *X* indicated.

Residual

This is the difference between the actual and predicted values of *Y*.

Standardized Residual

The variance of the observed residuals is not constant. This makes comparisons among the residuals difficult. One solution is to standardize the residuals by dividing by their standard deviations. This gives a set of residuals with constant variance.

The formula for this residual is

$$r_j = \frac{e_j}{s\sqrt{1 - h_{jj}}}$$

Percent Absolute Error

The percent is the absolute value of the *Residual* divided by the *Actual* value. Scrutinize observations with the large percent errors.

Residual Diagnostics Section

Residual Diagnostics Section

Row	Weight (X)	Residual	RStudent	Hat Diagonal	Cook's D	MSEi
1	159.0000	-1.8475	-1.3931	0.0607	0.0595	1.8723
2	155.0000	-2.0748	-1.5845	0.0567	0.0696	1.8176
3	157.0000	1.5389	1.1392	0.0586	0.0397	1.9381
4	125.0000	0.7203	0.5173	0.0560	0.0083	2.0537
5	103.0000	-3.0300	* -2.5957	0.0879	0.2462	1.4939
6	122.0000	-0.7002	-0.5034	0.0588	0.0083	2.0554
7	101.0000	1.3563	1.0150	0.0922	0.0522	1.9669
8	82.0000	1.0265	0.7818	0.1439	0.0525	2.0137
9	228.0000	-0.1761	* -0.1429	0.2712	0.0040	2.0836
10	199.0000	2.4258	* 2.0305	0.1499	0.3097	1.6789
11	195.0000	0.1985	0.1480	0.1369	0.0018	2.0834
12	110.0000	-0.3822	-0.2757	0.0748	0.0032	2.0768
13	191.0000	-1.0289	-0.7748	0.1248	0.0438	2.0149
14	151.0000	0.6979	0.5003	0.0537	0.0074	2.0558
15	119.0000	0.8793	0.6360	0.0620	0.0138	2.0376
16	119.0000	0.8793	0.6360	0.0620	0.0138	2.0376
17	112.0000	1.2315	0.9060	0.0716	0.0319	1.9900
18	87.0000	-0.9393	-0.7067	0.1283	0.0378	2.0265
19	190.0000	-0.8357	-0.6245	0.1219	0.0280	2.0393
20	87.0000	0.0607	0.0450	0.1283	0.0002	2.0858
21	100.0000			0.0944		
22	150.0000			0.0531		
23	200.0000			0.1533		
24						
25						
26						

Outliers are rows that are separated from the rest of the data. Influential rows are those whose omission results in a relatively large change in the results. This report lets you see both.

An outlier may be defined as a row in which |RStudent| > 2. A moderately influential row is one with a CooksD > 0.5. A heavily influential row is one with a CooksD > 1.

MSEi is the value of the Mean Square Error (the average of the sum of squared residuals) calculatedwith each row omitted.

This is a report gives residual diagnostics for each observation. These were discussed earlier in the technical of this chapter and we refer you to that section for the technical details.

X

This is the value of *X* at which the prediction is made.

Residual

This is the difference between the actual and predicted values of *Y*.

RStudent

Sometimes called the externally studentized residual, *RStudent* is a standardized residual that has the impact of a single observation removed from the mean square error. If the regression assumption of normality is valid, a single value of the RStudent has a *t* distribution with *N* - 2 degrees of freedom.

An observation is starred as an outlier if the absolute value of RStudent is greater than 2.

Hat Diagonal

The hat diagonal captures an observation's remoteness in the *X*-space. Some authors refer to the hat diagonal as a measure of *leverage* in the *X*-space.

Hat diagonals greater than 4 / *N* are considered influential. However, an influential observation is not a bad observation. An influential observation should be checked to determine if it is also an outlier.

Cook's D

Cook's D attempts to measure the influence the observation on all *N* fitted values. The formula for Cook's *D* is

$$D_j = \frac{\sum_{i=1}^{N} w_j \left[\hat{y}_j - \hat{y}_j(i)\right]^2}{ps^2}$$

The $\hat{y}_j(i)$ are found by removing observation *i* before the calculations. A Cook's *D* value greater than one indicates an observation that has large influence. Some statisticians have suggested that a better cutoff value is 4 / (*N* - 2).

MSEi

This is the value of the mean squared error calculated without observation *j*.

Leave One Row Out Section

Leave One Row Out Section

Row	RStudent	DFFITS	Cook's D	CovRatio	DFBETAS(0)	DFBETAS(1)
1	-1.3931	-0.3540	0.0595	0.9615	0.0494	-0.1483
2	-1.5845	-0.3885	0.0696	0.9023	0.0228	-0.1337
3	1.1392	0.2842	0.0397	1.0279	-0.0284	0.1087
4	0.5173	0.1260	0.0083	1.1511	0.0739	-0.0414
5	* -2.5957	-0.8059	0.2462	0.6304	-0.6820	0.5292
6	-0.5034	-0.1258	0.0083	1.1564	-0.0800	0.0486
7	1.0150	0.3234	0.0522	1.0978	0.2781	-0.2188
8	0.7818	0.3205	0.0525	1.2202	0.3024	-0.2589
9	-0.1429	-0.0872	0.0040	* 1.5346	0.0646	-0.0787
.	.	.	.	.	.	.
.	.	.	.	.	.	.
.	.	.	.	.	.	.

Each column gives the impact on some aspect of the linear regression of omitting that row.

RStudent represents the size of the residual. DFFITS represents the change in the fitted value of a row. Cook's D summarizes the change in the fitted values of all rows. CovRatio represents the amount of change in the determinant of the covariance matrix. DFBETAS(0) and DFBETAS(1) give the amount of change in the intercept and slope.

Each column gives the impact on some aspect of the linear regression of omitting that row.

RStudent

Sometimes called the externally studentized residual, *RStudent* is a standardized residual that has the impact of a single observation removed from the mean square error. If the regression assumption of normality is valid, a single value of the RStudent has a *t* distribution with *N* - 2 degrees of freedom.

An observation is starred as an outlier if the absolute value of RStudent is greater than 2.

Dffits

Dffits is the standardized difference between the predicted value of *Y* with and without observation *j*. It represents the number of estimated standard errors that the predicted value changes if that observation is omitted. Dffits > 1 would flag observations as being influential in prediction.

Cook's D

Cook's D attempts to measure the influence the observation on all *N* fitted values. The formula for Cook's *D* is

$$D_j = \frac{\sum_{i=1}^{N} w_j [\hat{y}_j - \hat{y}_j(i)]^2}{ps^2}$$

The $\hat{y}_j(i)$ are found by removing observation *i* before the calculations. A Cook's *D* value greater than one indicates an observation that has large influence. Some statisticians have suggested that a better cutoff value is 4 / (*N* - 2).

CovRatio

This diagnostic flags observations that have a major impact on the generalized variance of the regression coefficients. A value exceeding 1.0 implies that the observation provides an improvement, i.e., a reduction in the generalized variance of the coefficients. A value of CovRatio less than 1.0 flags an observation that increases the estimated generalized variance. This is not a favorable condition.

DFBETAS(0) and DFBETAS(1)

DFBETAS(0) and *DFBETAS(1)* are the standardized change in the intercept and slope when an observation is omitted from the analysis. Belsley, Kuh, and Welsch (1980) recommend using a cutoff of $2/\sqrt{N}$ when *N* is greater than 100. When *N* is less than 100, others have suggested using a cutoff of 1.0 or 2.0 for the absolute value of *DFBETAS*.

Outlier Detection Chart

Outlier Detection Chart

Row	Weight (X)	Residual		Standardized Residual		RStudent	
1	159.0000	-1.8475	|||||||||......	-1.3580	||||||||.......	-1.3931	|||||||........
2	155.0000	-2.0748	||||||||||.....	-1.5220	|||||||||......	-1.5845	|||||||||......
3	157.0000	1.5389	|||||||........	1.1299	|||||||........	1.1392	||||||.........
4	125.0000	0.7203	|||...........	0.5282	|||...........	0.5173	||.............
5	103.0000	-3.0300	||||||||||||||	-2.2604	||||||||||||||	* -2.5957	||||||||||||||
6	122.0000	-0.7002	|||...........	-0.5142	|||...........	-0.5034	||.............
7	101.0000	1.3563	||||||.........	1.0142	||||||.........	1.0150	|||||.........
8	82.0000	1.0265	|||||..........	0.7904	|||||..........	0.7818	||||...........
9	228.0000	-0.1761	|.............	-0.1470	|.............	-0.1429	|.............
10	199.0000	2.4258	|||||||||||....	1.8744	||||||||||||...	* 2.0305	|||||||||||....
11	195.0000	0.1985	|.............	0.1522	|.............	0.1480	|.............
12	110.0000	-0.3822	|.............	-0.2831	|.............	-0.2757	|.............
13	191.0000	-1.0289	||||..........	-0.7835	||||..........	-0.7748	||||..........
14	151.0000	0.6979	|||...........	0.5111	|||...........	0.5003	||.............
15	119.0000	0.8793	||||..........	0.6468	||||..........	0.6360	|||...........
16	119.0000	0.8793	||||..........	0.6468	||||..........	0.6360	|||...........
17	112.0000	1.2315	|||||..........	0.9105	|||||..........	0.9060	|||||..........
18	87.0000	-0.9393	||||..........	-0.7168	||||..........	-0.7067	|||...........
19	190.0000	-0.8357	|||...........	-0.6354	|||...........	-0.6245	|||...........
20	87.0000	0.0607	|.............	0.0463	|.............	0.0450	|.............
21	100.0000		|.............		|.............		|.............
22	150.0000		|.............		|.............		|.............
23	200.0000		|.............		|.............		|.............
24			|.............		|.............		|.............
25			|.............		|.............		|.............
26			|.............		|.............		|.............

Outliers are rows that are separated from the rest of the data. Since outliers can have dramatic effects on the results, corrective action, such as elimination, must be carefully considered. Outlying rows should not automatically be removed unless a good reason for their removal can be given.

An outlier may be defined as a row in which |RStudent| > 2. Rows with this characteristic have been starred.

Outliers are rows that are far removed from the rest of the data. Since outliers can have dramatic effects on the results, corrective action, such as elimination, must be carefully considered. Outlying rows should not be removed unless a good reason for their removal can be given.

An outlier may be defined as a row in which |RStudent| > 2. Rows with this characteristic have been starred.

X

This is the value of X.

Residual

This is the difference between the actual and predicted values of Y.

Standardized Residual

The variance of the observed residuals is not constant. This makes comparisons among the residuals difficult. One solution is to standardize the residuals by dividing by their standard deviations. This gives a set of residuals with constant variance.

RStudent

Sometimes called the externally studentized residual, *RStudent* is a standardized residual that has the impact of a single observation removed from the mean square error. If the regression assumption of normality is valid, a single value of the RStudent has a *t* distribution with *N* - 2 degrees of freedom.

An observation is starred as an outlier if the absolute value of RStudent is greater than 2.

Influence Detection Chart

Influence Detection Chart

Row	Weight (X)	DFFITS		Cook's D		DFBETAS(1)	
1	159.0000	-0.3540	||||||.........	0.0595	||............	-0.1483	||............
2	155.0000	-0.3885	||||||.........	0.0696	|||............	-0.1337	||............
3	157.0000	0.2842	||||..........	0.0397	|..............	0.1087	||............
4	125.0000	0.1260	|..............	0.0083	|..............	-0.0414	|..............
5	103.0000	-0.8059	||||||||||||||.	0.2462	||||||||||....	0.5292	||||||||||....
6	122.0000	-0.1258	|..............	0.0083	|..............	0.0486	|..............
7	101.0000	0.3234	|||||.........	0.0522	||............	-0.2188	||||..........
8	82.0000	0.3205	|||||.........	0.0525	||............	-0.2589	|||||.........
9	228.0000	-0.0872	|..............	0.0040	|..............	-0.0787	|..............
10	199.0000	0.8525	||||||||||||||||	0.3097	||||||||||||||	0.6959	||||||||||||||
11	195.0000	0.0589	|..............	0.0018	|..............	0.0470	|..............
12	110.0000	-0.0784	|..............	0.0032	|..............	0.0451	|..............
13	191.0000	-0.2925	||||..........	0.0438	||............	-0.2265	||||..........
14	151.0000	0.1192	|..............	0.0074	|..............	0.0312	|..............
15	119.0000	0.1635	||............	0.0138	|..............	-0.0720	|..............
16	119.0000	0.1635	||............	0.0138	|..............	-0.0720	|..............
17	112.0000	0.2515	||||..........	0.0319	|..............	-0.1381	||............
18	87.0000	-0.2712	||||..........	0.0378	|..............	0.2118	||||..........
19	190.0000	-0.2327	|||...........	0.0280	|..............	-0.1787	|||...........
20	87.0000	0.0173	|..............	0.0002	|..............	-0.0135	|..............
21	100.0000		|..............		|..............		|..............
22	150.0000		|..............		|..............		|..............
23	200.0000		|..............		|..............		|..............
24			|..............		|..............		|..............
25			|..............		|..............		|..............
26			|..............		|..............		|..............

Influential rows are those whose omission results in a relatively large change in the results. They are not necessarily harmful. However, they will distort the results if they are also outliers. The impact of influential rows should be studied very carefully. Their accuracy should be double-checked.
DFFITS is the standardized change in Yhat when the row is omitted. A row is influential when DFFITS > 1 for small datasets (N < 30) or when DFFITS > 2*SQR(1/N) for medium to large datasets.

Cook's D gives the influence of each row on the Yhats of all the rows. Cook suggests investigating all rows having a Cook's D > 0.5. Rows in which Cook's D > 1.0 are very influential.

DFBETAS(1) is the standardized change in the slope when this row is omitted. DFBETAS(1) > 1 for small datasets (N < 30) and DFBETAS(1) > 2/SQR(N) for medium and large datasets are indicative of influential rows.

Influential rows are those whose omission results in a relatively large change in the results. They are not necessarily harmful. However, they will distort the results if they are also outliers. The impact of influential rows should be studied very carefully. The accuracy of the data values should be double-checked.

X

This is the value of *X*.

Dffits

Dffits is the standardized difference between the predicted value of *Y* with and without observation *j*. It represents the number of estimated standard errors that the predicted value changes if that observation is omitted. Dffits > 1 would flag observations as being influential in prediction.

Cook's D

Cook's D attempts to measure the influence the observation on all *N* fitted values. The formula for Cook's *D* is

$$D_j = \frac{\sum_{i=1}^{N} w_j \left[\hat{y}_j - \hat{y}_j(i)\right]^2}{ps^2}$$

The $\hat{y}_j(i)$ are found by removing observation *i* before the calculations. A Cook's *D* value greater than one indicates an observation that has large influence. Some statisticians have suggested that a better cutoff value is 4 / (*N* - 2).

DFBETAS(1)

DFBETAS(1) is the standardized change in the slope when an observation is omitted from the analysis. Belsley, Kuh, and Welsch (1980) recommend using a cutoff of $2/\sqrt{N}$ when *N* is greater than 100. When *N* is less than 100, others have suggested using a cutoff of 1.0 or 2.0 for the absolute value of *DFBETAS*.

Outlier and Influence Chart

Outlier & Influence Chart

Row	Weight (X)	RStudent (Outlier)		Cooks D (Influence)		Hat Diagonal (Leverage)	
1	159.0000	-1.3931	||||||.......	0.0595	||............	0.0607	|..............
2	155.0000	-1.5845	||||||||......	0.0696	|||...........	0.0567	|..............
3	157.0000	1.1392	|||||........	0.0397	|..............	0.0586	|..............
4	125.0000	0.5173	||............	0.0083	|..............	0.0560	|..............
5	103.0000	* -2.5957	|||||||||||||	0.2462	||||||||||....	0.0879	|..............
6	122.0000	-0.5034	||............	0.0083	|..............	0.0588	|..............
7	101.0000	1.0150	|||||.........	0.0522	||............	0.0922	|..............
8	82.0000	0.7818	||||...........	0.0525	||............	0.1439	|||...........
.	.	.	. .	.	. .	.	. .
.	.	.	. .	.	. .	.	. .
.	.	.	. .	.	. .	.	. .

Outliers are rows that are separated from the rest of the data. Influential rows are those whose omission results in a relatively large change in the results. This report lets you see both.

An outlier may be defined as a row in which |RStudent| > 2. A moderately influential row is one with a CooksD > 0.5. A heavily influential row is one with a CooksD > 1.

This report provides diagnostics about whether a row is an outlier, influential, and has high leverage. Outliers are rows that are removed from the rest of the data. Influential rows are those whose omission results in a relatively large change in the results. This report lets you see both.

X

This is the value of *X*.

RStudent (Outlier)

RStudent is a standardized residual that has the impact of a single observation removed from the mean square error. If the regression assumption of normality is valid, a single value of the RStudent has a *t* distribution with *N* - 2 degrees of freedom.

An observation is starred as an outlier if the absolute value of RStudent is greater than 2.

Cook's D (Influence)

Cook's D attempts to measure the influence the observation on all *N* fitted values. The formula for Cook's *D* is

$$D_j = \frac{\sum_{i=1}^{N} w_j [\hat{y}_j - \hat{y}_j(i)]^2}{ps^2}$$

The $\hat{y}_j(i)$ are found by removing observation *i* before the calculations. A Cook's *D* value greater than one indicates an observation that has large influence. Some statisticians have suggested that a better cutoff value is 4 / (*N* - 2).

Hat Diagonal (Leverage)

The hat diagonal captures an observation's remoteness in the *X*-space. Some authors refer to the hat diagonal as a measure of *leverage* in the *X*-space.

Hat diagonals greater than 4 / *N* are considered influential. However, an influential observation is not a bad observation. An influential observation should be checked to determine if it is also an outlier.

Inverse Prediction of X Means

Inverse Prediction of X Means

Row	Height (Y)	Weight (X)	Predicted Weight (Xhat|Y)	X-Xhat|Y	Lower 95% Conf. Limit of X Mean|Y	Upper 95% Conf. Limit of X Mean|Y
1	64.0000	159.0000	149.4360	9.5640	145.9832	153.0193
2	63.0000	155.0000	144.2591	10.7409	140.8441	147.7361
3	67.0000	157.0000	164.9664	-7.9664	161.1310	169.1387
4	60.0000	125.0000	128.7287	-3.7287	125.1181	132.1948
5	52.0000	103.0000	87.3141	15.6859	81.4894	92.4444
6	58.0000	122.0000	118.3750	3.6250	114.3947	122.0735
7	56.0000	101.0000	108.0214	-7.0214	103.5227	112.1007
8	52.0000	82.0000	87.3141	-5.3141	81.4894	92.4444
9	79.0000	228.0000	227.0884	0.9116	219.7388	235.5997
10	76.0000	199.0000	211.5579	-12.5579	205.2283	218.8430
11	73.0000	195.0000	196.0274	-1.0274	190.6564	202.1477
12	56.0000	110.0000	108.0214	1.9786	103.5227	112.1007
13	71.0000	191.0000	185.6737	5.3263	180.8886	191.0708
14	65.0000	151.0000	154.6128	-3.6128	151.0743	158.3507
15	59.0000	119.0000	123.5518	-4.5518	119.7777	127.1129
16	59.0000	119.0000	123.5518	-4.5518	119.7777	127.1129
17	58.0000	112.0000	118.3750	-6.3750	114.3947	122.0735
18	51.0000	87.0000	82.1372	4.8628	75.9418	87.5695
19	71.0000	190.0000	185.6737	4.3263	180.8886	191.0708
20	52.0000	87.0000	87.3141	-0.3141	81.4894	92.4444
21		100.0000				
22		150.0000				
23		200.0000				
24	50.0000		76.9604		70.3836	82.7054
25	60.0000		128.7287		125.1181	132.1948
26	70.0000		180.4969		175.9821	185.5549

This confidence interval estimates the mean of X in a large sample of individuals with this value of Y. This method of inverse prediction is also called 'calibration.'

This report provides inverse prediction or calibration results. Although a regression of *Y* on *X* has been fit, our interest here is predicting the value of *X* from the value of *Y*. This report provides both a point estimate and an interval estimate of the predicted mean of *X* given *Y*.

Y

This is the actual value of *Y*.

X

This is the value of *X* at which the prediction is made.

Predicted X (Xhat|Y)

The predicted value of *X* for the value of *Y* indicated.

Lower 95% Confidence Limit of X Mean|Y

This is the lower limit of a 95% confidence interval estimate of the mean of *X* at this value of *Y*.

Upper 95% Confidence Limit of X Mean|Y

This is the upper limit of a 95% confidence interval estimate of the mean of *X* at this value of *Y*.

Inverse Prediction of X Individuals

Inverse Prediction of X Individuals

Row	Height (Y)	Weight (X)	Predicted Weight (Xhat|Y)	X-Xhat|Y	Lower 95% Prediction Limit of X|Y	Upper 95% Prediction Limit of X|Y
1	64.0000	159.0000	149.4360	9.5640	133.7858	165.2167
2	63.0000	155.0000	144.2591	10.7409	128.5906	159.9896
3	67.0000	157.0000	164.9664	-7.9664	149.3036	180.9662
4	60.0000	125.0000	128.7287	-3.7287	112.9365	144.3765
5	52.0000	103.0000	87.3141	15.6859	70.7003	103.2335
6	58.0000	122.0000	118.3750	3.6250	102.4436	134.0246
7	56.0000	101.0000	108.0214	-7.0214	91.9059	123.7175
8	52.0000	82.0000	87.3141	-5.3141	70.7003	103.2335
9	79.0000	228.0000	227.0884	0.9116	210.4214	244.9172
10	76.0000	199.0000	211.5579	-12.5579	195.2744	228.7969
11	73.0000	195.0000	196.0274	-1.0274	180.0432	212.7609
12	56.0000	110.0000	108.0214	1.9786	91.9059	123.7175
13	71.0000	191.0000	185.6737	5.3263	169.8391	202.1203
14	65.0000	151.0000	154.6128	-3.6128	138.9697	170.4553
15	59.0000	119.0000	123.5518	-4.5518	107.6957	139.1949
16	59.0000	119.0000	123.5518	-4.5518	107.6957	139.1949
17	58.0000	112.0000	118.3750	-6.3750	102.4436	134.0246
18	51.0000	87.0000	82.1372	4.8628	65.3728	98.1386
19	71.0000	190.0000	185.6737	4.3263	169.8391	202.1203
20	52.0000	87.0000	87.3141	-0.3141	70.7003	103.2335
21		100.0000				
22		150.0000				
23		200.0000				
24	50.0000		76.9604		60.0352	93.0537
25	60.0000		128.7287		112.9365	144.3765
26	70.0000		180.4969		164.7214	196.8156

This prediction interval estimates the predicted value of X for a single individual with this value of Y. This method of inverse prediction is also called 'calibration.'

This report provides inverse prediction or calibration results. Although a regression of *Y* on *X* has been fit, our interest here is predicting the value of *X* from the value of *Y*. This report provides both a point estimate and an interval estimate of the predicted value of *X* given *Y*.

Y

This is the actual value of *Y*.

X

This is the value of *X* at which the prediction is made.

Predicted X (Xhat|Y)

The predicted value of *X* for the value of *Y* indicated.

Lower 95% Prediction Limit of X|Y

This is the lower limit of a 95% prediction interval estimate of *X* at this value of *Y*.

Upper 95% Prediction Limit of X|Y

This is the upper limit of a 95% prediction interval estimate of *X* at this value of *Y*.